APPLIED MECHANICS

STATICS

by

GEORGE W. HOUSNER

AND

DONALD E. HUDSON

Division of Engineering
California Institute of Technology

SEVENTH PRINTING

D. VAN NOSTRAND COMPANY, INC.

PRINCETON, NEW JERSEY

TORONTO — LONDON

NEW YORK

D. VAN NOSTRAND COMPANY, INC.
120 Alexander St., Princeton, New Jersey (*Principal office*)
257 Fourth Avenue, New York 10, New York

D. Van Nostrand Company, Ltd.
358, Kensington High Street, London, W.14, England

D. Van Nostrand Company (Canada), Ltd.
25 Hollinger Road, Toronto 16, Canada

Published simultaneously in Canada by
D. Van Nostrand Company (Canada), Ltd.

First Published July 1949
Reprinted January 1951, January 1954, August 1955
December 1956, January 1958,
August 1958

PRINTED IN THE UNITED STATES OF AMERICA

PREFACE

This book has been prepared as a textbook for the use of Sophomore Engineering students. The book may thus be said to be on an intermediate level; it has been planned not only to give the student certain factual information which he will need in his profession, but also to form a logical transition from the elements of mechanics as studied in the general physics course to the more advanced courses in engineering.

As a textbook, the main emphasis has been on method and on the development of fundamental principles, so that the book is not to be considered as a treatise or a reference work on the subject. The fundamental principles, however, have been illustrated by a number of applications to important practical problems drawn from the various engineering fields. It is intended that the examples should not only clarify the principles but also indicate the very broad applicability of the principles and methods of mechanics to all of the fields of applied science.

Although the book begins with the simplest elements of mechanics, it is believed that some advantage can be taken of the fact that the student will have had a previous introduction to the subject in his Freshman physics course. This does not mean that it can be assumed that the student is a master of even the simplest elements of the subject. It does mean, however, that such topics as the vector representation of a force, the addition of planar forces, and the elementary concept of a moment can be considered as review, and that such subjects can be immediately treated as simple cases of more general propositions.

The conventional separation between Statics and Dynamics has been retained. Although in some respects it is more logical to develop Statics as a special case of Dynamics, no practical advantage is gained by such an arrangement, particularly for those students who have made a logical survey of mechanics in their general physics course. This book is followed by a com-

panion volume on Dynamics, which is used by the students at the California Institute of Technology during the second half of their Junior year, after they have taken a course in Theory of Elasticity and Strength of Materials.

It is believed by the authors that the essential simplicity of the subject of Statics has often been obscured in the minds of some students by the complexity of the applications and examples which have been used to illustrate the basic principles. In an effort to avoid this difficulty the treatment of Statics has been arranged in two parts. The first part, consisting of the first three chapters, gives a concise treatment of the general principles of statics. The examples and applications in these first three chapters are intended to clarify the principles rather than to indicate the possible range of application or to provide the student with sufficient practice in using the principles. The second section includes a large number of applications of Statics to the various engineering fields, such as Structures, Fluid Mechanics, and Machine Design. This section should not only give the student experience in applying the principles of statics, but it should also indicate the unity which exists between the various engineering subjects.

It is realized that the treatment of many of the applications has been somewhat superficial, and that the student will make a more complete analysis of such topics as fluid statics and thick-film lubrication later in more specialized courses. It is believed, however, that such brief discussions may combat the tendency towards a too rigorous compartmentalization of courses, which may prevent the student from attaining that broad general view of applied science which is so necessary. In all such cases care has been taken to use methods which can be extended later for the more complete treatment, and the student has been informed of the limitations of the analysis.

Some topics, such as static stability and the Principle of Virtual Displacements, which have been included, are not ordinarily treated in an elementary text. These have been introduced to show the student that he has not exhausted the subject and to suggest further lines of development for the abler student. An attempt has been made to view the subject from several different

standpoints. Mechanics is a growing science, and there is no reason to suppose that new formulations of basic laws may not be made, which may, for certain problems, be superior to the older methods. It would be unfortunate if any student were to suppose that Engineering Mechanics is in its complete and final form—devised once and for all by some superhuman intelligence in the past and delivered to us entire, to be accepted on faith.

One of the definite objectives of the Statics course is the development in the student of skill in the formulation and numerical solution of problems. The only way in which most students can develop this skill is to solve a large number of problems. When time is available the instructor will wish to supplement the problems in the text with additional problems suited to the interests and abilities of the students.

The authors wish to express their appreciation to Dr. F. C. Lindvall, Chairman of the Division of Engineering, California Institute of Technology, for the assistance he has given and the interest he has taken in the development of this book. Thanks are also expressed to Mr. W. L. Johnson and Mr. Seba Eldridge, Jr., for the assistance they have rendered in the preparation of problems. Finally, several of the translations of quotations which appear at the heads of the chapters were taken from *A Source Book in Physics*, Magie, W. F., McGraw-Hill Book Co., 1935.

G. W. H.
D. E. H.

Pasadena, Calif.
January, 1949

CONTENTS

CHAPTER I

THE COMPOSITION AND RESOLUTION OF FORCE SYSTEMS

Any number of forces are always reducible to a single force and a single couple whose plane is perpendicular to the direction of the force. So that there is always in space a certain determinate straight line . . . which may serve at the same time to represent the line of action of the resultant, and the axis of the resultant couple . . . this central axis . . . gives us so luminous a reduction of all of the forces of the system as to throw a light at the same time upon all the other equivalent reductions, and to group them, so to speak, into a single picture, in which we see at once order and mutual dependence.—L. Poinsot, *Éléments de Statique* (1803).

Mechanics is the science of motion and has as its object the investigation of the position and configurations of systems of material bodies as a function of time. The aim of this investigation is to arrive at the simplest description of the facts which are observed in nature and, from this description, to form generalizations which will permit valid predictions as to the behavior of other systems.

The motions that occur in nature are the result of interactions between the various bodies which make up the system under consideration. That portion of the subject of mechanics which describes the motion of bodies, without reference to the causes of the motion, is called *kinematics*. That portion of the subject which studies the relationship between the mutual influences and the resulting motions is called *kinetics*. The subjects of kinematics and kinetics are often considered together under the name of *dynamics*. That portion of the subject which studies the conditions under which the interactions between bodies counteract each other, with a resultant state of rest or of uniform motion, is called *statics*. When the mutual influences between the bodies of a system counteract each other so that the system is either at rest or moves with a uniform velocity, the system is said to be in *equilibrium*. Statics, then, is the study of the conditions of equilibrium. In a large number of engineering problems the structures or mechan-

isms which are under consideration are in equilibrium, so that the subject of statics assumes an important place in the field of applied mechanics.

The Concept of a Force. The mutual influences between bodies, which are responsible for the motions of the bodies, are called *forces*. The idea of a force can be formed only with reference to the consequence of the action of the force upon a material body. The action of a force upon a body may be manifested in two different ways, as can be seen from the following example.

Consider the two situations illustrated in Fig. 1–1. A mass rests on a smooth horizontal plane. In (a) it is free to move

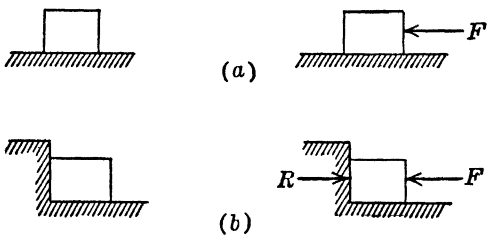


Fig. 1–1

either to the left or right whercas in (b) it is constrained so that it cannot move to the left. We now apply the force F as shown and note the results in each instance. In (a) the mass acquires an acceleration equal to the force divided by the mass, but in (b) the *active force* F is opposed by a *reactive force* R, in such a way that $R = -F$ and equilibrium of the system results. Thus in case (a) the effect of the force has been to change the motion of the mass, while in case (b) the effect of the force has been to set up a reactive force which maintains the system in equilibrium. The effect of the two forces R and F in case (b) on any actual body would be to cause a deformation of the body, in this case a compression. We may thus say that there are two effects of the action of a force on a material body—a motional effect and a deformational effect.

Either one of these effects could be used to define and to measure a force and in practice both are used. A common method of measuring the magnitude of a force is to measure the deformation which the force will set up in a calibrated spring

balance. This method clearly makes use of the deformational effect of the force, since the system as a whole is in equilibrium.

It is also customary to define and to measure a force by means of its motional effects. The relation between the force exerted on a body and the motion which results is given by Newton's Laws, which may be taken as a definition of a force. For our present purpose in statics, however, we need only emphasize that aspect of a force which we imply when we say "a force is a push or a pull," and which is thus directly related to our muscular sense.

To specify completely the action of a force upon a body, the magnitude, direction, and point of application of the force must be given. A force may thus be represented by a vector which gives the magnitude and direction of the force and which indicates, by its location, the line of action and the point of application of the force.

The Composition and Resolution of Forces. When a number of forces act upon a body they form a *force system.* By the *resultant force* of a system we mean the vector sum of the forces of the system. To specify completely the motional effect of a system of forces, the line of action of the resultant force vector must also be known. The vector sum can be determined by any of the several methods for adding vector quantities. Analytical methods will be used for the most part in the present book, but graphical methods will be used when appropriate.

Consider a system of two forces in equilibrium. This situation can occur only if the two forces lie along the same straight line and are equal in magnitude and opposite in direction (Fig. 1–2a).

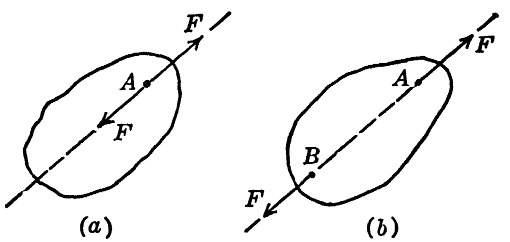


Fig. 1–2

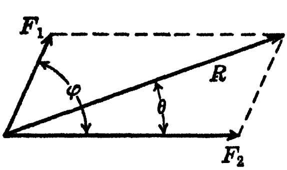
Suppose the system of Fig. 1–2a is changed so that the two forces which originally acted at the same point A now act at two different points, A and B, on the line of action of the two forces.

The system is still in equilibrium so far as the motion of the body is concerned, but the deformations of the system will have been altered. In general we mean by a force system in *equilibrium*, one whose resultant motional effect on a material body is zero. We thus conclude that the force F in Fig. 1–2b can be moved along its line of action from A to B without changing the conditions of equilibrium of the system. This conclusion can be given in the form of the *principle of transmissibility*, which states that a force can be considered to act at any point along its line of action, so far as the conditions of equilibrium are concerned.

A simplification often introduced into mechanics is the idea of a *rigid body*, defined as a body in which every particle maintains fixed distances from all of the other particles of the body. It follows that there can be no deformation of a rigid body, and only the motional effects of force systems which act upon a rigid body need to be considered. For rigid bodies, the principle of transmissibility is seen to be completely true, and in this sense the principle is more properly considered as a property of a rigid body than as a property of a force. The principles of statics, however, as they relate to systems in equilibrium and to the motional effects of forces, need not be restricted to rigid bodies. In fact, in many engineering problems in which the equations of equilibrium are used, it will be necessary to account for the deformations of the system in order to arrive at a complete solution.

In many problems it is desired to replace a single force by two or more forces which are equivalent to it. This problem occurs, for example, when a force may be exerted in a certain direction only, while it is desired to have a resultant force in some other direction. The two or more forces which replace the given force are called *components* of the force, and the process of finding the components is called the *resolution* of a force into components.

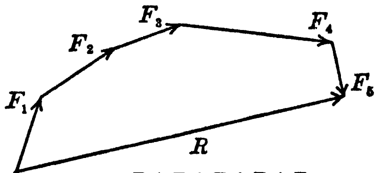
PROBLEMS



PROB. 1

1. Vector quantities may be added by means of the parallelogram law, as shown in the figure. Write the expression for the magnitude and direction of the resultant of the two forces F_1 and F_2, and show that R vanishes only when $F_1 = F_2$ and $\varphi = 180°$.

2. By successive applications of the parallelogram law, show that a system of any number of forces can be added by the polygon law, as shown in the figure, and show that the resultant is independent of the order in which the forces are added.

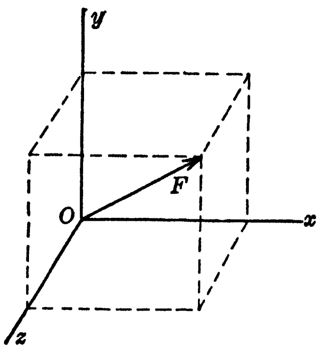


$=F_1+F_2+F_3+F_4+F_5$

PROB. 2

3. The polygon law as shown in Problem 2 above, permits the determination of the resultant force, but the line of action of the resultant is not located. Show how the line of action of the resultant force may be located by using the principle of the transmissibility of forces and the parallelogram law.

4. The line of action of a force F passes through the origin of an x, y, z rectangular coordinate system. Show that the components of the force along the three coordinate axes are: $F_x = F \cos \theta_x$, $F_y = F \cos \theta_y$, $F_z = F \cos \theta_z$, and $F = \sqrt{F_x^2 + F_y^2 + F_z^2}$ where θ_x, θ_y, θ_z are the angles between the line of action of the force and the coordinate axes.



PROB. 4

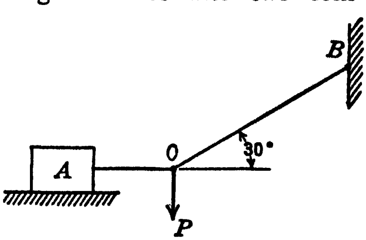
5. Show graphically by means of the polygon law that the sum of the components of a system of forces along a particular line is equal to the component, along that line, of the resultant of the system of forces.

6. Show how to resolve a given force graphically into two components having given magnitudes.

7. Show how to resolve a given force graphically into two components, one of which has a given magnitude, while the other has a given direction.

8. Show how to resolve graphically a given force into two components, one of which has a given magnitude and direction.

9. It is desired to exert a horizontal force of 25 lb on the body A by means of a vertical pull P as shown in the diagram. Find the magnitude of P and the magnitude of the force in the cable OB.



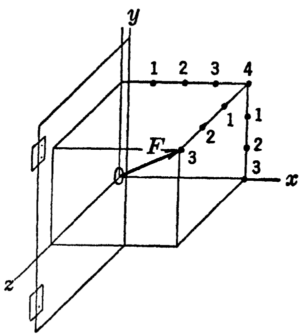
PROB. 9

10. Find the resultant of the five forces specified by the coordinates given in the following table:

	x	y	z
120 lb	−4	4	−2
220	6	1	1
90	3	−2	2
80	0	0	1
100	−1	0	0

The beginning of each force vector is at the origin of the rectangular coordinate system. The coordinates given above locate points on the lines of action of the forces.

11. A force F is applied to a doorknob as shown in the figure. If it requires a force of 20 lb perpendicular to the door to open it, what should be the magnitude of F?

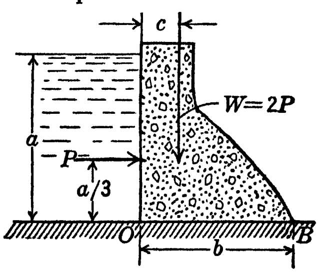


Prob. 11

12. A force of 10 lb which makes an angle of 30° with the x-axis is to be resolved into three components in the plane of the forces. One of these components has a magnitude of 10 lb, and a direction $\theta_x = 120°$; another component has the magnitude 5 lb and the direction $\theta_x = 60°$. Find the magnitude and direction of the third component.

13. A force of 50 lb acting along the x-axis is to be resolved into three components in a plane. One of these components has the magnitude 20 lb and makes an angle $\theta_x = 30°$ with the x-axis. The directions of the other two components are $\theta_x = 240°$ and $\theta_x = 300°$. Find the magnitudes of the components.

14. The figure represents a section through a gravity type masonry dam. The water force P is equal to $(62.4)(a)\left(\frac{a}{2}\right) = 31.2a^2$ and the dam weight, W, is equal to $2P$. The cross-sectional shape of the dam is such that $c = 0.2a$. If the dam base is $b = 0.625a$, find the distance x from O, at which the resultant of the forces acting on the dam intersects the base OB, and find the resultant force. Ordinarily the resultant force in a gravity type dam should intersect the base within the middle third of the base width. The forces given are the forces acting upon a unit length of the dam, perpendicular to the paper.



Prob. 14

Systems of Parallel Forces. The resultant force vector for a system of parallel forces can be found in the usual way by vector addition. The line of action of this resultant, however, cannot be determined by the direct application of the parallelogram law. The method of locating the resultant is shown in Fig. 1–3 for two parallel forces.

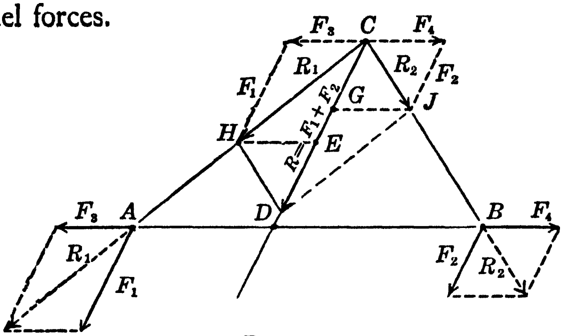


FIG. 1–3

The given forces F_1 and F_2 are applied at the points A and B and have parallel lines of action. Suppose that two equal and opposite forces F_3 and F_4 are added to the system in such a way that their common line of action passes through the points A and B, and that the forces F_3 and F_4 act at the points A and B respectively. Since these two forces F_3 and F_4 form a system in equilibrium, their addition will in no way change the conditions of equilibrium of the original system. It is now possible to combine F_1 and F_3, and F_2 and F_4 by means of the parallelogram law, and hence to replace the two original parallel forces F_1 and F_2 by the two non-parallel forces R_1 and R_2. The two forces R_1 and R_2 can then be combined to find the line of action of the resultant of the system.

From Fig. 1–3, an analytical method of locating the resultant of two parallel forces can be developed. From the geometry of the system we have the following relations:

$$\triangle ACD \text{ is similar to } \triangle CEH$$

$$\therefore \frac{AD}{CD} = \frac{F_3}{F_1}; \quad (F_3)(CD) = (F_1)(AD)$$

$$\triangle BCD \text{ is similar to } \triangle CGJ$$

$$\therefore \frac{BD}{CD} = \frac{F_3}{F_2}; \quad (F_3)(CD) = (F_2)(BD)$$

thus:

$$(F_1)(AD) = (F_2)(BD) \tag{1}$$

$$\frac{AD}{BD} = \frac{F_2}{F_1}$$

This expression gives an analytical method of determining the location of the resultant R.

There is one type of parallel force system which has a particularly important property. This is a system of two parallel forces that have equal magnitudes but opposite directions. For this system the vector sum of the two forces vanishes, and there is no single resultant force which is equivalent to the system. It will be seen in Fig. 1–4 that the graphical method of Fig. 1–3 also fails to give a solution of a single force for a system of this kind. We thus conclude that a pair of equal, opposite, parallel and non-colinear forces cannot be reduced to a single resultant force, but is already in its simplest form. Such a system of forces is called a *couple*.

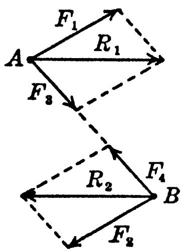


Fig. 1–4

The physical effect of a couple acting on a body is to produce a rotation of the body. Since we need a measure of the tendency of a couple to cause rotation, we are led to the concept of a moment or torque, which is discussed in the following paragraph.

The Definition of a Moment. Consider a force F located a distance a from the origin O of a rectangular coordinate system (Fig. 1–5); a is measured in the plane of the force F and the point O, perpendicular to the line of action of F. The *moment* of F about the point O, or about the axis through O normal to the plane of F and O, is defined as the vector M, whose magnitude is $(F)(a)$, whose direction is the line OA, normal to the plane of F and a, and whose sense is the direction of advance of a right-handed screw rotated by the force F. This rule for sense may also be stated by saying that if

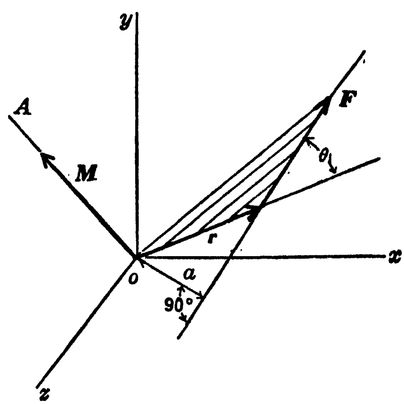


Fig. 1–5

one looks in the direction of M, F will appear to turn clockwise around M.

The Theorem of Moments (Varignon's Theorem). It can be shown that for any force system the moment of the resultant about any point or axis is equal to the sum of the moments of the individual forces about that point or axis. We shall demonstrate this theorem for the special case of two co-planar, non-parallel forces.

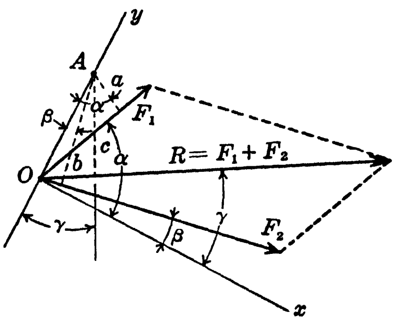


FIG. 1–6

Consider the two forces F_1 and F_2 shown in Fig. 1–6, with point of concurrence O and moment-axis A. Take for the y-axis a line through O and A. Writing components along the x-axis, we have:

$$R \cos \gamma = F_1 \cos \alpha + F_2 \cos \beta$$

Multiply each term by the distance OA.

$$R(OA) \cos \gamma = F_1(OA) \cos \alpha + F_2(OA) \cos \beta$$

From the geometry of Fig. 1–6, it will be seen that:

$$(OA) \cos \alpha = a$$
$$(OA) \cos \beta = b$$
$$(OA) \cos \gamma = c$$

so that:

$$R(c) = F_1(a) + F_2(b)$$

Since $(F_1)(a)$ is the moment of F_1 about A, $(F_2)(b)$ is the moment of F_2 about A, and $(R)(c)$ the moment of the resultant about A, the theorem for this case is proved.

If a moment vector M is defined with respect to a point O, then its component in any direction OD is the moment about the line OD. In Fig. 1–5, using the theorem of moments, the components of M along the three coordinate axes can be determined by resolving the force F into its rectangular components

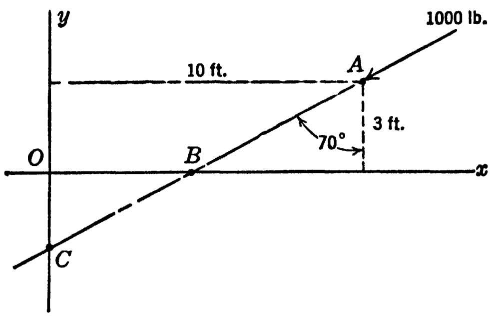
F_x, F_y, and F_z and writing the sums of the moments of these components about the coordinate axes. This gives:

$$M_x = yF_z - zF_y$$
$$M_y = zF_x - xF_z$$
$$M_z = xF_y - yF_x$$

The signs of the terms have been fixed by the definition of a moment vector as previously given. If one looks from the origin towards the positive end of the coordinate axes, a clockwise rotation means a positive sign.

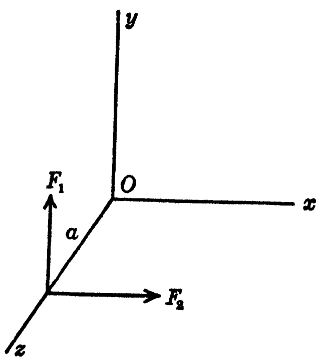
PROBLEMS

15. Prove the theorem of moments for a system of two parallel forces. The diagram of Fig. 1–3 and the results of the accompanying analysis may be used.



PROB. 16

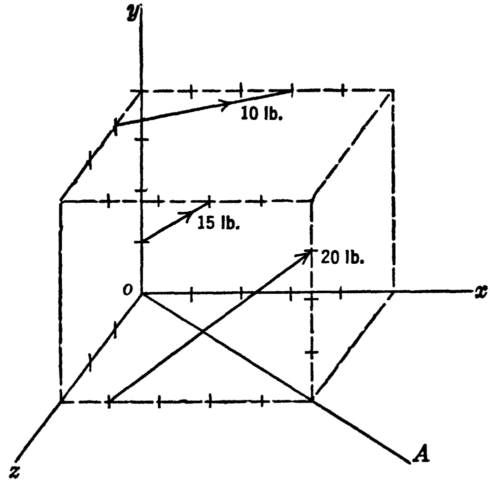
16. Calculate the moment of the 1000-lb force about the point O, (a) by multiplying the force by the perpendicular distance from O to the line of action of the force, and (b) by resolving the force into rectangular components at the point A, using the theorem of moments. (c) Repeat, by resolving the force into components at points B and C.



PROB. 17

17. F_1 is parallel to the y-axis and F_2 is parallel to the x-axis and the lines of action of the two forces intersect the z-axis at a distance a from the xy plane. Show that if the two vectors M_1 and M_2 representing the moments of F_1 and F_2 about O are added, the resultant vector M correctly represents the moment of the resultant of the two forces about the point O.

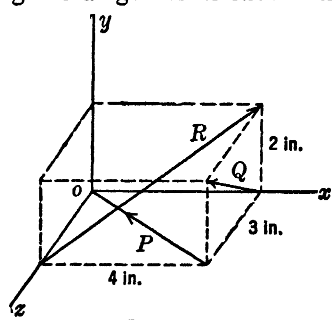
18. Find the moment of the three forces shown in the figure about the axis OA. Do this by first finding the components of the moments



Prob. 18

along the three coordinate axes, and then by adding the components of these components along the line OA.

19. Forces P, Q, and R are applied along the diagonals as shown in the figure. If $2P = 2Q = R = 1000$ lb, what is the moment sum about each coordinate axis? What is the resultant moment about O?



Prob. 19

20. Given $F = 600$ lb, $F_x = M_x = 0$. The point 3, 4, 12 is a point on the line of action of F. Find M_y and M_z.

21. A plane force system is found to have: $\Sigma F_x = 100$ lb, $\Sigma F_y = -160$ lb, $\Sigma M_0 = 500$ ft-lb. Determine the point at which the resultant force intersects the x-axis.

22. The x, y, and z components of a force are -20, 50, and 25 lb. The moments are $M_x = 400$, $M_y = 180$, $M_z = -40$ ft-lb. Locate the point of intersection of the force with the xz plane.

The Product of Two Vectors. The definition of a moment as the product of a force times a distance is equivalent to stating that it is the product of two vectors, a force vector and a displacement vector.

The method of adding or subtracting vectors has been defined by the parallelogram law, and multiplication of a vector by a scalar presents no new problems, it being necessary only to multiply the magnitude of the vector by the scalar and to leave the direction unchanged. When we come to the product of two vectors, however, there are several possible definitions which might be used. Two vector products are commonly employed and are of importance in the field of mechanics.

The Scalar or Dot Product. Given two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ at an angle θ (Fig. 1–7),* the scalar product of the two vectors is defined as a scalar quantity having the magnitude $(a)(b) \cos \theta$. The scalar product is thus equal to the magnitude of one vector multiplied by the component of the second vector in the direction of the first vector. The scalar product is written $\boldsymbol{a} \cdot \boldsymbol{b}$, with a dot between the two vectors, and hence it is often called the dot product.

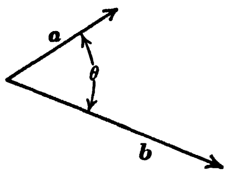


Fig. 1–7

The same magnitude of the scalar product will result whether $\boldsymbol{a}$ is multiplied by the component of $\boldsymbol{b}$ in the direction of $\boldsymbol{a}$, or $\boldsymbol{b}$ is multiplied by the component of $\boldsymbol{a}$ in the direction of $\boldsymbol{b}$, since:

$$(a)(b \cos \theta) = (b)(a \cos \theta)$$

so we may say:

$$\boldsymbol{a} \cdot \boldsymbol{b} = \boldsymbol{b} \cdot \boldsymbol{a}$$

i.e., scalar multiplication is a commutative process.

If the angle between the two vectors is greater than 90°, but less than 270°, the scalar product will be negative.

The most common example of scalar products in mechanics is in connection with the work done by a force. If a force $\boldsymbol{F}$ of constant magnitude has its point of application moved through some displacement $\boldsymbol{S}$ then the work done by the force is defined as the scalar quantity which is equal to the displacement multi-

* Where special attention is to be directed to the fact that a quantity is a vector, the symbol will be printed in boldface type. Certain common quantities such as forces and moments, which are always to be understood to be vector quantities, may be printed in either boldface or ordinary type. Any operation, such as addition or multiplication, involving vector quantities is always to be understood to be a vector operation.

plied by the component of the force in the direction of the displacement, hence:

$$\text{Work} = \boldsymbol{F} \cdot \boldsymbol{S}$$

We shall make use of the concept of work in later chapters.

The Analytical Description of a Vector. The description of a vector by means of its rectangular components is a convenient method, as we have already seen in the particular case of force vectors. The conventions with respect to a rectangular coordinate system may be seen in Fig. 1–8. The *unit vectors* $\boldsymbol{i}$, $\boldsymbol{j}$, $\boldsymbol{k}$ are defined as vectors having magnitudes of unity and the directions of the x, y, z coordinate axes respectively. The vector component along one of the coordinate axes can thus be indicated by giving the magnitude of the component with the proper unit vector. For example, in Fig. 1–8, the x-component of the vector $\boldsymbol{a}$ would be written as $a_x\boldsymbol{i}$ and the vector $\boldsymbol{a}$ would be written as $\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}$. We may write the scalar product of two vectors as follows:

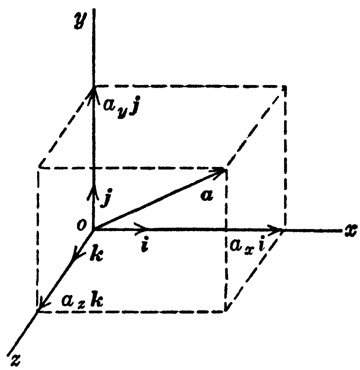


FIG. 1–8

$$\boldsymbol{a} = a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}$$
$$\boldsymbol{b} = b_x\boldsymbol{i} + b_y\boldsymbol{j} + b_z\boldsymbol{k}$$
$$\begin{aligned}\boldsymbol{a} \cdot \boldsymbol{b} &= (a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}) \cdot (b_x\boldsymbol{i} + b_y\boldsymbol{j} + b_z\boldsymbol{k}) \\ &= a_xb_x\boldsymbol{i} \cdot \boldsymbol{i} + a_xb_y\boldsymbol{i} \cdot \boldsymbol{j} + a_xb_z\boldsymbol{i} \cdot \boldsymbol{k} \\ &\quad + a_yb_x\boldsymbol{j} \cdot \boldsymbol{i} + a_yb_y\boldsymbol{j} \cdot \boldsymbol{j} + a_yb_z\boldsymbol{j} \cdot \boldsymbol{k} \\ &\quad + a_zb_x\boldsymbol{k} \cdot \boldsymbol{i} + a_zb_y\boldsymbol{k} \cdot \boldsymbol{j} + a_zb_z\boldsymbol{k} \cdot \boldsymbol{k}\end{aligned}$$

From the definition of a scalar product, we have the following relations between the unit vectors:

$$\boldsymbol{i} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{j} = \boldsymbol{k} \cdot \boldsymbol{k} = 1$$

since in each case the vectors have magnitudes of 1 and are in the same direction. Also:

$$\boldsymbol{i} \cdot \boldsymbol{j} = \boldsymbol{j} \cdot \boldsymbol{i} = \boldsymbol{i} \cdot \boldsymbol{k} = \boldsymbol{k} \cdot \boldsymbol{i} = \boldsymbol{j} \cdot \boldsymbol{k} = \boldsymbol{k} \cdot \boldsymbol{j} = 0$$

since the two vectors of the respective sets are perpendicular. The dot product is, therefore,

$$\boldsymbol{a} \cdot \boldsymbol{b} = a_xb_x + a_yb_y + a_zb_z$$

The Vector or Cross Product. In Fig. 1–9 are shown two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, which make an angle $\theta < 180°$ measured in the plane of the two vectors. The vector product $\boldsymbol{a} \times \boldsymbol{b}$ is defined as a vector along line OB normal to the plane of the vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, with such a direction that, if one looks in the direction OB, a clockwise turn will bring the direction of $\boldsymbol{a}$ into that of $\boldsymbol{b}$. The rule for direction may also be stated by saying that the direction is that of the advance of a right-handed screw turned from $\boldsymbol{a}$ to $\boldsymbol{b}$. The magnitude of the vector product is defined as $(a)(b) \sin\theta$. The magnitude is thus equal to twice the area of the triangle shown shaded in Fig. 1–9. Because of the form in which it is written the vector product $\boldsymbol{a} \times \boldsymbol{b}$ is often called the cross product.

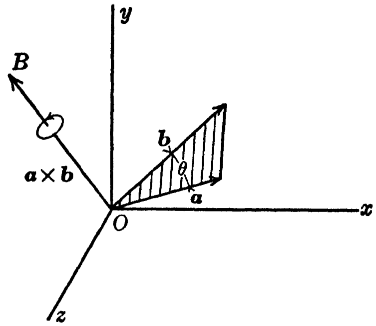


FIG. 1–9

From the above definition it is seen that the vector product depends upon the order in which the vectors are taken. The vector product $\boldsymbol{b} \times \boldsymbol{a}$ would have the same magnitude as $\boldsymbol{a} \times \boldsymbol{b}$, but would have the opposite direction so that

$$\boldsymbol{b} \times \boldsymbol{a} = -\boldsymbol{a} \times \boldsymbol{b}$$

i.e., vector products are not commutative.

Writing the vector product in the form of components, we have:

$$\begin{aligned}\boldsymbol{a} \times \boldsymbol{b} &= (a_x\boldsymbol{i} + a_y\boldsymbol{j} + a_z\boldsymbol{k}) \times (b_x\boldsymbol{i} + b_y\boldsymbol{j} + b_z\boldsymbol{k}) \\ &= a_xb_x\boldsymbol{i} \times \boldsymbol{i} + a_xb_y\boldsymbol{i} \times \boldsymbol{j} + a_xb_z\boldsymbol{i} \times \boldsymbol{k} \\ &+ a_yb_x\boldsymbol{j} \times \boldsymbol{i} + a_yb_y\boldsymbol{j} \times \boldsymbol{j} + a_yb_z\boldsymbol{j} \times \boldsymbol{k} \\ &+ a_zb_x\boldsymbol{k} \times \boldsymbol{i} + a_zb_y\boldsymbol{k} \times \boldsymbol{j} + a_zb_z\boldsymbol{k} \times \boldsymbol{k}\end{aligned}$$

From the definition of the vector product, we have the following relations between the unit vectors $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$:

$$\begin{aligned}&\boldsymbol{i} \times \boldsymbol{i} = \boldsymbol{j} \times \boldsymbol{j} = \boldsymbol{k} \times \boldsymbol{k} = 0 \\ &\boldsymbol{i} \times \boldsymbol{j} = -\boldsymbol{j} \times \boldsymbol{i} = \boldsymbol{k} \\ &\boldsymbol{i} \times \boldsymbol{k} = -\boldsymbol{k} \times \boldsymbol{i} = -\boldsymbol{j} \\ &\boldsymbol{j} \times \boldsymbol{k} = -\boldsymbol{k} \times \boldsymbol{j} = \boldsymbol{i}\end{aligned}$$

hence:

$$\boldsymbol{a} \times \boldsymbol{b} = (a_y b_z - a_z b_y)\boldsymbol{i} + (a_z b_x - a_x b_z)\boldsymbol{j} + (a_x b_y - a_y b_x)\boldsymbol{k}$$

The Moment of a Force as a Vector Product. Returning to our consideration of the moment of a force, and referring to Fig. 1–5, we now see that we can write the moment vector as:

$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F}$$

By the definition of the vector product, this statement is equivalent to the definition of a moment given previously. The vector $\boldsymbol{r}$ joins the point about which the moment is to be taken and any point on the line of action of the force.

The components of the vector $\boldsymbol{M}$ can be written as follows:

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$
$$\boldsymbol{F} = F_x\boldsymbol{i} + F_y\boldsymbol{j} + F_z\boldsymbol{k}$$
$$\boldsymbol{M} = \boldsymbol{r} \times \boldsymbol{F} = (yF_z - zF_y)\boldsymbol{i} + (zF_x - xF_z)\boldsymbol{j} + (xF_y - yF_x)\boldsymbol{k}$$

so that the components of $\boldsymbol{M}$ are:

$$M_x = yF_z - zF_y$$
$$M_y = zF_x - xF_z$$
$$M_z = xF_y - yF_x$$

These are, of course, the same expressions obtained from Fig. 1–5. The introduction of the vector product makes possible a concise notation for moment vectors. Note that the sign convention for moments previously mentioned is consistent with the sign convention for vector products.

PROBLEMS

23. Show that two vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ are perpendicular if $\boldsymbol{a} \cdot \boldsymbol{b} = 0$. What is the significance of $\boldsymbol{a} \times \boldsymbol{b} = 0$?

24. Show that both the scalar product and the vector product obey the distributive law of ordinary multiplication, i.e.,

$$(\boldsymbol{a} + \boldsymbol{b}) \cdot \boldsymbol{c} = \boldsymbol{a} \cdot \boldsymbol{c} + \boldsymbol{b} \cdot \boldsymbol{c}$$
$$(\boldsymbol{a} + \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{c} + \boldsymbol{b} \times \boldsymbol{c}$$

25. Given two vectors

$$\boldsymbol{a} = 3\boldsymbol{i} + 2\boldsymbol{j} + 5\boldsymbol{k}$$
$$\boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j}$$

Find the following vectors:

$$\boldsymbol{a} + \boldsymbol{b}; \quad \boldsymbol{a} - \boldsymbol{b}; \quad \boldsymbol{a} \cdot \boldsymbol{b}; \quad \boldsymbol{a} \times \boldsymbol{b}$$

26. Show that the vector product can be written as

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

27. Given two vectors

$$\boldsymbol{a} = 3\boldsymbol{i} + 2\boldsymbol{j} + \boldsymbol{k}$$
$$\boldsymbol{b} = \boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$$

Find the angle between these vectors, using the definition of a scalar product.

28. Find the moment about the point $\boldsymbol{i} + 3\boldsymbol{j} + 2\boldsymbol{k}$ of a force represented by the vector from the point $2\boldsymbol{i} + \boldsymbol{j} - \boldsymbol{k}$ to the point $2\boldsymbol{i} - \boldsymbol{k}$ at a scale of 10 lb per ft.

29. A body is rotating about an axis with an angular speed of ω radians per second. In vector notation $\boldsymbol{\omega}$ designates the angular velocity with magnitude ω and with direction of the axis of rotation given by the right-hand screw rule. If $\boldsymbol{v}$ is the linear velocity of a point A in the above-mentioned body and $\boldsymbol{r}$ is the position vector of the point A with respect to any point of the rotation axis, show that,

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$$

30. A body is rotating with an angular speed of 2 radians per second about an axis parallel to $3\boldsymbol{j} - 4\boldsymbol{k}$ passing through the point $2\boldsymbol{i} + \boldsymbol{k}$ ft. Referring to the result of the preceding problem, express vectorially the velocity of the point $3\boldsymbol{i} + 2\boldsymbol{k}$ ft.

The Characteristics of a Couple. A couple has been defined as a system of two parallel forces equal in magnitude but opposite in direction. We shall now show that the sum of the moments of the forces of a couple is independent of the point about which the moments are taken.

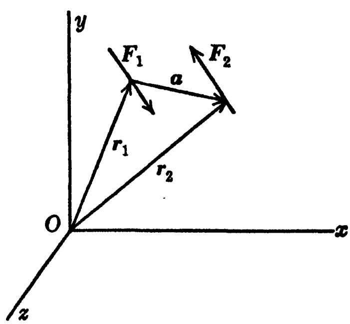


FIG. 1–10

Consider two equal and opposite forces $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$; $\boldsymbol{F}_1 + \boldsymbol{F}_2 = 0$ (Fig. 1–10). Let $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ be vectors drawn from any point O to points on the lines of action of $\boldsymbol{F}_1$ and $\boldsymbol{F}_2$. $\boldsymbol{a}$ is the vector $(\boldsymbol{r}_2 - \boldsymbol{r}_1)$. The sum of the moments of the two forces of the couple will be called the *moment of the couple* and will be represented by the vector $\boldsymbol{C}$.

$$\boldsymbol{C} = \boldsymbol{r}_1 \times \boldsymbol{F}_1 + \boldsymbol{r}_2 \times \boldsymbol{F}_2$$

Since $F_1 = -F_2$, this becomes:

$$C = (r_2 - r_1) \times F_2 = a \times F_2$$

Since O can be taken as any point without altering the vector a, the moment of the couple is independent of the center about which the moments are taken. The moment is thus a property of the couple itself and can be taken as a quantitative measure of the magnitude of the couple. A couple, therefore, is specified completely by the couple vector $C = a \times F$. The magnitude of the couple vector is equal to $(F)(a \sin \theta) = (F)(d)$, where d is the perpendicular distance between the forces of the couple. The direction is normal to the plane containing the two forces and the sense corresponds to the direction of advance of a right-handed screw rotated by the forces of the couple.

Since C is independent of any particular point, the couple vector does not have a unique line of action as does a force vector.

From the above definition of a couple as a vector, a number of characteristics follow directly:

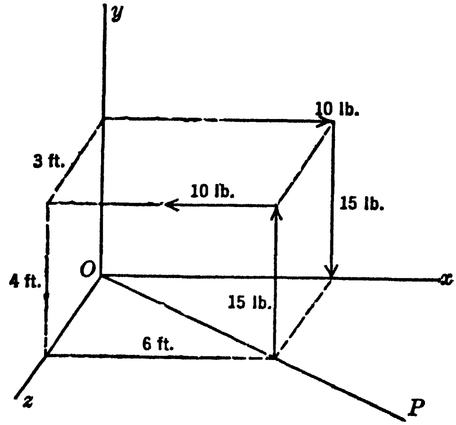
1. Two couples are equivalent (have the same motional effect upon a body) if their moments and directions are the same. The particular values of the force and the moment arm are not significant; it is only the product of the two which determines the action of the couple.
2. The forces of a couple can be rotated through any angle in their plane, or translated to any position in the plane, without changing the motional effect of the couple upon a body.
3. The forces of a couple can be translated into any parallel plane without changing the motional effect of the couple upon a body.

PROBLEMS

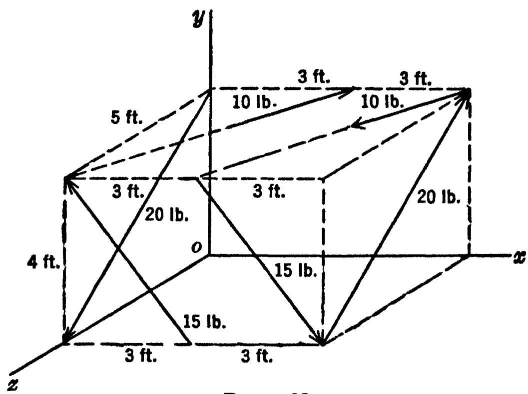
31. Given the two couples as shown in the diagram, find their sum: (a) By adding the couple forces directly and thus forming a new couple. (b) By adding the couple vectors.

32. Referring to the diagram in the preceding problem, find the sum of the moments of the forces of the system about the line OP which lies in the xz plane. Do this by first taking moments directly about the

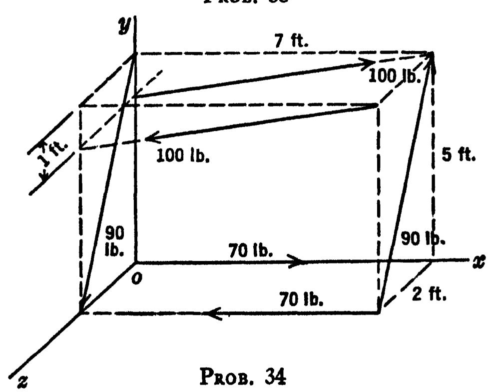
line *OP*, and then by finding the component of the resultant couple vector, as found in the preceding problem, along the line *OP*.



PROB. 31



PROB. 33

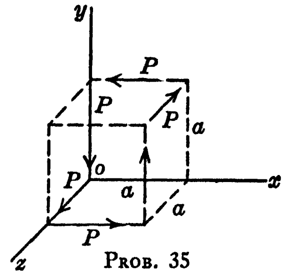


PROB. 34

33. Find the resultant of the system of three couples shown in the diagram.

34. Find the resultant couple of the three space couples shown in the figure.

35. Six forces of equal magnitude act along the edges of a cube as shown in the diagram. Find the resultant of the system.



PROB. 35

The Resolution of a Force into a Force and a Couple. The action of a single force can always be duplicated by an equal parallel force plus a couple. Suppose that it is desired to resolve the force $\boldsymbol{F}$ of Fig. 1–11a, whose point of action is B, into a parallel

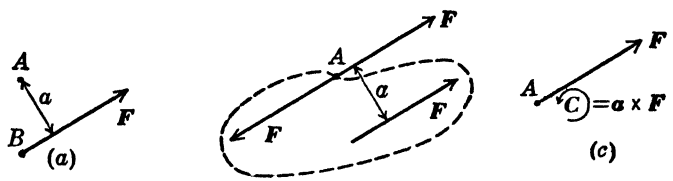


FIG. 1–11

force $\boldsymbol{F}$ passing through the point A, plus a couple. We introduce into the system the two equal and opposite forces $\boldsymbol{F}$ and $-\boldsymbol{F}$ acting at the point A. Since these two forces form a system in equilibrium they can be added without altering the motional effects of the original $\boldsymbol{F}$ upon the system. The circled forces in Fig. 1–11b can now be treated as a couple $\boldsymbol{C} = \boldsymbol{a} \times \boldsymbol{F}$, and the original single force has been resolved into an equal parallel force and a couple.

The Resultant of the General Force System. A general analytical method can now be developed for determining the resultant of any three-dimensional force system. The method consists of replacing each force of the system by an equivalent force passing through a particular point and a couple. These forces and couples may then easily be added to give the resultant force and the resultant couple for the whole system. The method is indicated in Fig. 1–12 where, for simplicity, two forces only are shown. The method is valid, however, for any number of forces.

In Fig. 1–12a the two general forces F_1 and F_2 are given and the problem is to determine their resultant. We resolve each

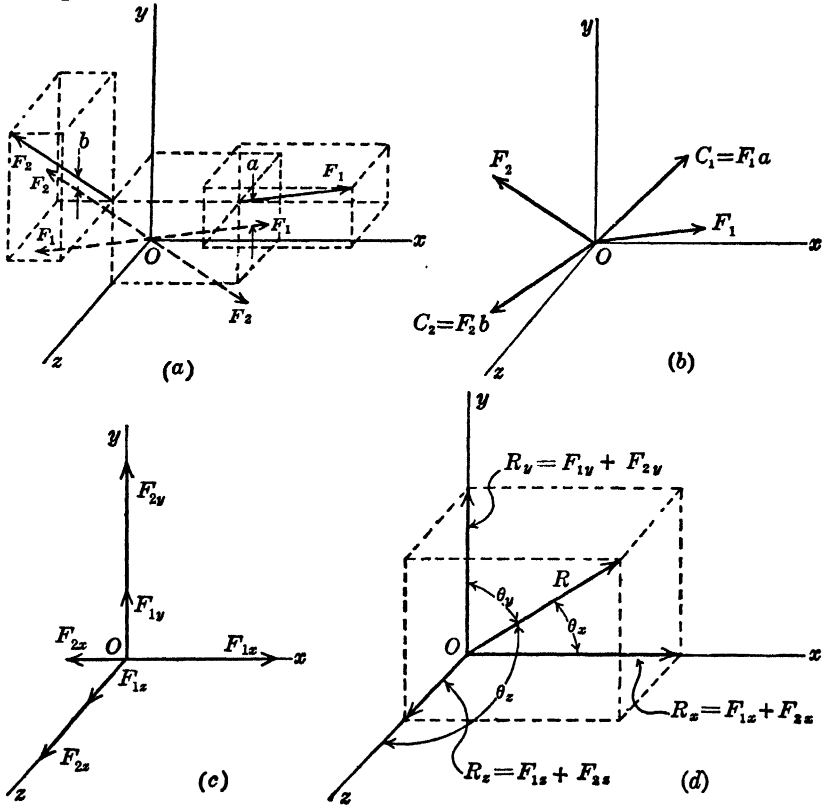


Fig. 1–12

force into a parallel force, through the origin of a rectangular coordinate system, and a couple C, thus reducing the system to a set of concurrent forces and couples as shown in (b). Each of the concurrent forces can be resolved into components along the three coordinate axes as in (c), and the sum of these rectangular components will give the rectangular components of the resultant force R which can then be found as in (d). The couple vectors can be added in exactly the same way, and hence the general force system is reduced to a force and a couple given by:

$$R = \sqrt{R_x^2 + R_y^2 + R_z^2};$$

$$\theta_x = \cos^{-1}\frac{R_x}{R}, \quad \theta_y = \cos^{-1}\frac{R_y}{R}, \quad \theta_z = \cos^{-1}\frac{R_z}{R}$$

where $$R_x = \Sigma F_x; \quad R_y = \Sigma F_y; \quad R_z = \Sigma F_z$$

and $$C = \sqrt{C_x^2 + C_y^2 + C_z^2};$$

$$\phi_x = \cos^{-1} \frac{C_x}{C}, \quad \phi_y = \cos^{-1} \frac{C_y}{C}, \quad \phi_z = \cos^{-1} \frac{C_z}{C}$$

where $$C_x = \Sigma M_x; \quad C_y = \Sigma M_y; \quad C_z = \Sigma M_z$$

In practice, it is not necessary to imagine that the forces have been resolved into forces and couples, as in Fig. 1–12. Since the forces at the origin will always be parallel to the corresponding forces in their original position, the rectangular components of the forces in their original system will be the same as the rectangular coordinates of the concurrent force system. In the case of the couples, one of the forces of each couple passes through the origin and hence has a zero moment about the three coordinate axes. The moment of each of the original forces about one of the coordinate axes will thus equal the component of the couple vector along that axis. We need only consider the general force system in its given position, and write:

$$\begin{aligned} R_x &= \Sigma F_x & \quad R &= \sqrt{R_x^2 + R_y^2 + R_z^2} \\ R_y &= \Sigma F_y & \quad \theta_x &= \cos^{-1} \frac{R_x}{R}, \text{ etc.} \\ R_z &= \Sigma F_z & & \\ C_x &= \Sigma M_x & \quad C &= \sqrt{C_x^2 + C_y^2 + C_z^2} \\ C_y &= \Sigma M_y & \quad \phi_x &= \cos^{-1} \frac{C_x}{C}, \text{ etc.} \\ C_z &= \Sigma M_z & & \end{aligned}$$

This method reduces any general three-dimensional force system to a single force and a single couple. For most problems this is as far as it is necessary to reduce such force systems. In fact, in most engineering problems, the components of the force and the couple along the coordinate axes are wanted, so that usually it will not be necessary to calculate the resultant force and moment.

A system of a force and a couple can always be resolved into a force passing through a particular point and a couple whose vector is parallel to the force, i.e., the couple vector can be made parallel to the force vector. The axis along which the force vector acts in this case is called the *central axis* of the system.

The analysis of force systems along the lines outlined above was first made by Louis Poinsot (1777–1859) and treated in his

two principal works, the *Éléments de Statique* (1803) and the *Nouvelle Théorie de la Rotation des Corps* (1834). In these works, along with other important contributions to dynamics, he developed the theory of couples and of the central axis and, for the first time, systematized the solution of general force systems.

EXAMPLE 1. Find the resultant of the force system shown in Fig. 1–13.

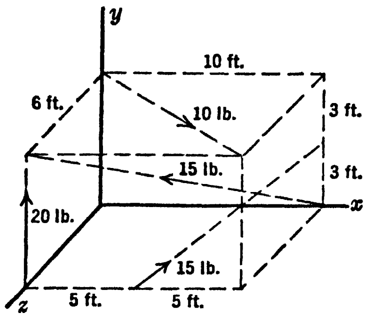


FIG. 1–13

Solution.

$$R_x = \Sigma F_x = +(10)\frac{10}{\sqrt{(6)^2+(10)^2}} - (15)\frac{10}{\sqrt{(6)^2+(6)^2+(10)^2}} + (15)\frac{5}{\sqrt{(3)^2+(5)^2+(6)^2}}$$
$$= +8.58 - 11.45 + 8.96 = 6.09 \text{ lb}$$

$$R_y = \Sigma F_y = +20 + (15)\frac{6}{\sqrt{(6)^2+(6)^2+(10)^2}} + (15)\frac{3}{\sqrt{(3)^2+(5)^2+(6)^2}}$$
$$= +20.0 + 6.87 + 5.38 = 32.25 \text{ lb}$$

$$R_z = \Sigma F_z = +(10)\frac{6}{\sqrt{(6)^2+(10)^2}} + (15)\frac{6}{\sqrt{(6)^2+(6)^2+(10)^2}} - (15)\frac{6}{\sqrt{(3)^2+(5)^2+(6)^2}}$$
$$= +5.15 + 6.87 - 10.76 = 1.26 \text{ lb}$$

Thus the resultant force is the vector:

$$\mathbf{R} = 6.09\mathbf{i} + 32.25\mathbf{j} + 1.26\mathbf{k} \text{ lb}$$

The magnitude and direction of this vector can be found if desired:

$$R = \sqrt{(6.09)^2 + (32.25)^2 + (1.26)^2} = 32.8 \text{ lb}$$

$$\theta_x = \cos^{-1} \frac{6.09}{32.8} = \cos^{-1} 0.1852 = 79.3°$$

$$\theta_y = \cos^{-1} \frac{32.25}{32.8} = \cos^{-1} 0.9840 = 10.3°$$

$$\theta_z = \cos^{-1} \frac{1.26}{32.8} = \cos^{-1} 0.0384 = 87.8°$$

$$C_x = \Sigma M_x = -(20)(6) + \frac{(10)(6)(6)}{\sqrt{(6)^2 + (10)^2}} - \frac{(15)(3)(6)}{\sqrt{(3)^2 + (5)^2 + (6)^2}}$$

$$= -120 + 31.2 - 32.3 = -121.1 \text{ ft-lb}$$

$$C_y = \Sigma M_y = \frac{-(15)(6)(10)}{\sqrt{(6)^2 + (6)^2 + (10)^2}} + \frac{(15)(6)(10)}{\sqrt{(3)^2 + (5)^2 + (6)^2}}$$

$$= -68.7 + 107.6 = 38.9 \text{ ft-lb}$$

$$C_z = \Sigma M_z = \frac{+(15)(10)(6)}{\sqrt{(6)^2 + (6)^2 + (10)^2}} + \frac{(15)(3)(5)}{\sqrt{(3)^2 + (5)^2 + (6)^2}} - \frac{(10)(10)(6)}{\sqrt{(6)^2 + (10)^2}}$$

$$= +68.7 + 26.9 - 51.5 = 44.1 \text{ ft-lb}$$

and

$$\boldsymbol{C} = -121.1\boldsymbol{i} + 38.9\boldsymbol{j} + 44.1\boldsymbol{k} \text{ ft-lb}$$

$$C = \sqrt{(121.1)^2 + (38.9)^2 + (44.1)^2} = 135 \text{ ft-lb}$$

$$\phi_x = \cos^{-1} \frac{-121.1}{135} = \cos^{-1}(-0.898) = 153.9°$$

$$\phi_y = \cos^{-1} \frac{38.9}{135} = \cos^{-1} 0.288 = 73.3°$$

$$\phi_z = \cos^{-1} \frac{44.1}{135} = \cos^{-1} 0.327 = 70.9°$$

The resultant force and couple are shown in Fig. 1–14.

Example 2. Find the resultant of the system of parallel forces shown in Fig. 1–15.

Solution. We note that for any parallel force system the forces of the resultant couple can be arranged parallel to the resultant force, and hence that the resultant couple and the resultant force

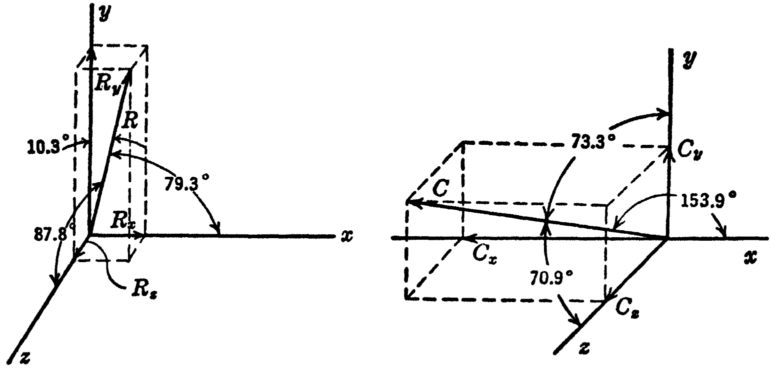


FIG. 1-14

can always be resolved into a single force or a single couple. Instead of applying the general method to this problem, there-

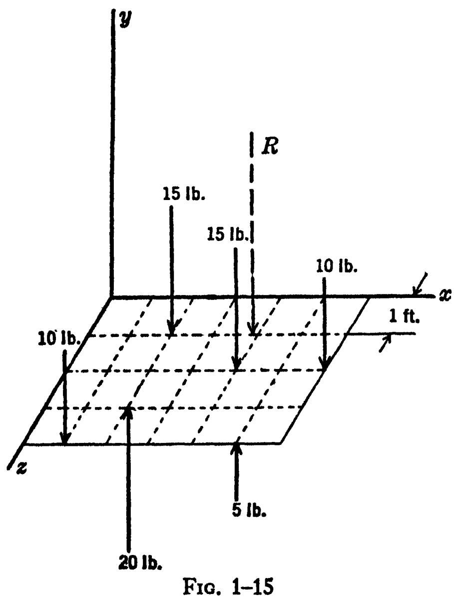


FIG. 1-15

fore, we shall use the principle of moments directly to find the location of the resultant force:

$$R = \Sigma F = -10 - 15 - 15 - 10 + 5 + 20 = -25 \text{ lb}$$

The distance z of the resultant from the xy plane can be found by taking moments about the x-axis.

$$(R)(z) = \Sigma M_x = (1)(15) + (2)(15) + (2)(10) + (4)(10) - (3)(20) - (4)(5)$$
$$= 15 + 30 + 20 + 40 - 60 - 20 = +25 \text{ ft-lb}$$

The downward resultant must be in such a position as to give a positive moment about the x-axis.

$$z = \frac{\Sigma M_x}{R} = \frac{25 \text{ ft-lb}}{25 \text{ lb}} = 1 \text{ ft in front of the } xy \text{ plane}$$

The distance x of the resultant from the yz plane can be found by taking moments about the z-axis.

$$(R)(x) = \Sigma M_z = -(1)(10) + (2)(20) - (2)(15) - (4)(15) + (5)(5) - (6)(10)$$
$$= -10 + 40 - 30 - 60 + 25 - 60 = -95 \text{ ft-lb}$$

The downward resultant must be in such a position as to give a negative moment about the z-axis.

$$x = \frac{\Sigma M_z}{R} = \frac{95 \text{ ft-lb}}{25 \text{ lb}} = 3.8 \text{ ft to the right of the } yz \text{ plane}$$

It should be observed that in general it is not possible to define the sign of a moment by taking the product of the sign of the moment arm and the sign of the force.

In Fig. 1–16 are seen two forces F which have the same direction of moment about the x-axis. In each case the direction of the moment is counterclockwise as one looks from the origin towards the positive end of the x-axis; hence by the right-hand screw rule the moments are negative. For the force parallel to the y-axis, the force is positive and the moment arm is positive, so that the product is positive. For the force parallel to the z-axis, the force is negative and the moment arm is positive, hence the product is negative. It will thus be apparent that no sign convention will give entire consistency between directions of forces, arms, and moments. The use of vector notation eliminates any difficulty of this kind.

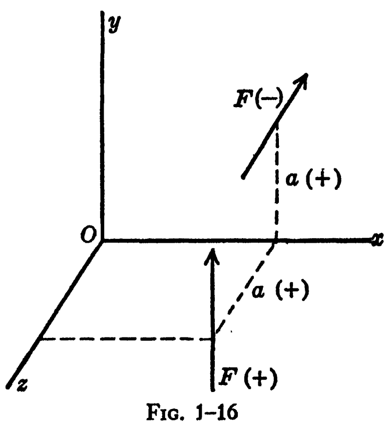
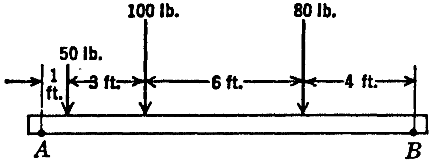


FIG. 1–16

PROBLEMS

36. Three forces act on a beam as shown in the diagram. (a) Find the resultant of the system. (b) Resolve this resultant into two components at the points A and B.



Prob. 36

37. Five forces which are parallel to the y-axis intersect the xz plane at the following points:

$$F_1 = 100 \text{ lb}, x = 2, z = 3; \quad F_2 = 150 \text{ lb}, x = 1, z = -1$$
$$F_3 = 80 \text{ lb}, x = -3, z = 5; \quad F_4 = -50 \text{ lb}, x = 3, z = 3$$
$$F_5 = -100 \text{ lb}, x = 5, z = 2$$

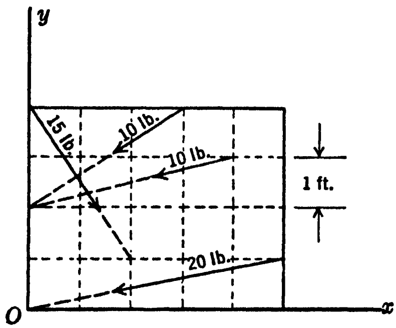
Find the resultant of the system.

38. A force of 50 lb acts vertically downward, parallel to the y-axis, and intersects the xz plane at the point $x = 2$, $z = 3$. Resolve this force into three parallel components acting at the points $x = 0$, $z = 0$; $x = 5$, $z = 0$; $x = 0$, $z = 4$.

39. Find the resultant of the force system shown in the diagram.

40. Find the resultant of the three concurrent forces shown in the figure.

41. Show that a general system of forces all of which lie in one plane can be reduced either to a single force or a single couple.

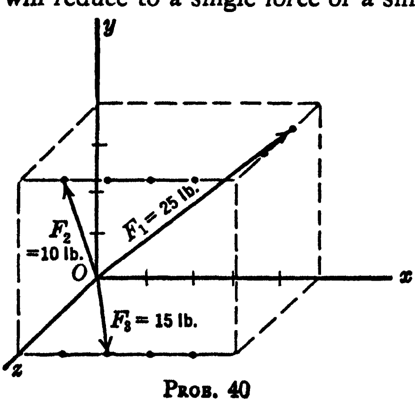


Prob. 39

42. Show that the condition that a general non-coplanar force system will reduce to a single force or a single couple is:

$$R_xC_x + R_yC_y + R_zC_z = 0$$

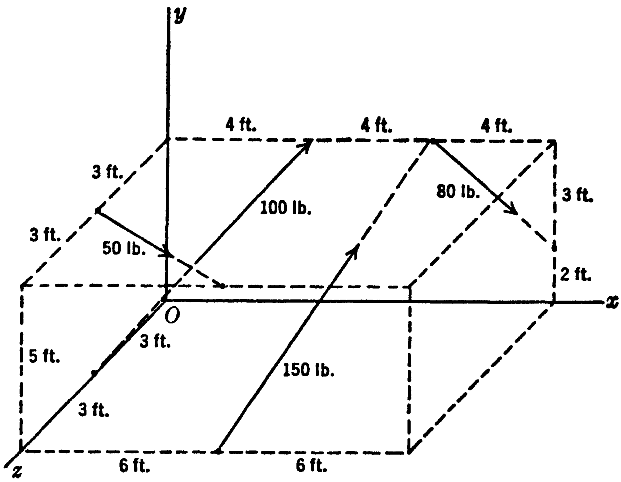
From this condition show that a general parallel system or a general coplanar system can be reduced to either a single force or a single couple.



Prob. 40

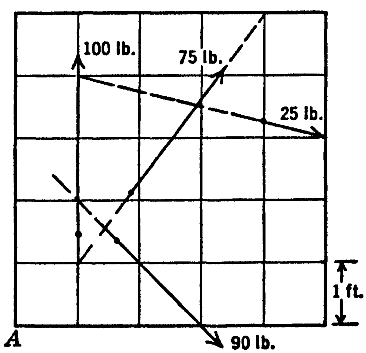
43. Show that a force and a couple can always be resolved into a force and a couple whose vector is parallel to the force vector. Show that in this case the couple has its smallest possible value.

44. Find the resultant of the forces shown in the diagram.

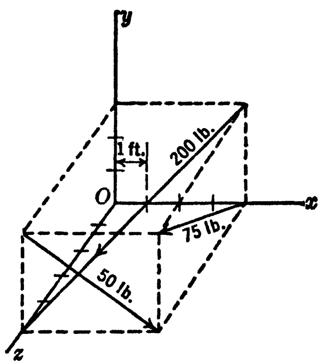


PROB. 44

45. Determine the resultant of the forces shown in the diagram. Find a single force through the point *A* and a couple which will together be equivalent to the given system.



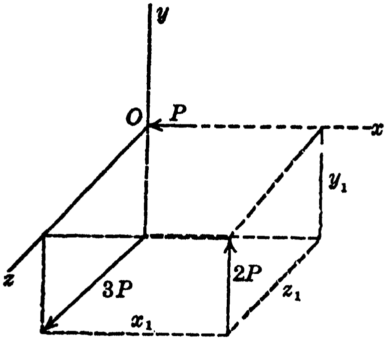
PROB. 45



PROB. 46

46. Replace the three forces shown in the figure by a single force acting through the origin and a couple.

47. The three forces shown in the figure are P, $2P$, and $3P$ as indicated. State the algebraic relationship between the dimensions x_1, y_1, and z_1 so that a single force may be the equivalent of the given forces.



PROB. 47

The Isolation of a Mechanical System — Free-Body Diagrams. From the preceding discussion it is seen that the methods of treating a system of given forces are relatively straightforward. The problems which confront the engineer or physicist, however, usually come in the form of some physical situation, so that the first step is to determine the system of forces which will represent the physical situation correctly. The application of the principles of statics to the forces of a system can often be reduced to a routine procedure. A more difficult problem is that of deciding which forces should be included in a particular system in order to arrive at a desired result. Skill in the manipulation of force systems will not suffice to solve the physical problem unless the forces which are equivalent to the given physical situation are included in the problem. On the other hand, once the true force picture is set up, the solutions practically always can be obtained by conventional methods which though they may involve extensive labor, ordinarily will require no great amount of ingenuity. The transition from a physical situation to a statement of the forces involved is the important first step in the solution of a problem.

Although the steps involved in recognizing the forces which must be included are not exactly the same for all problems, there are certain general principles which can be observed. The most important principle is that one must always distinguish between the particular mechanical system which is being investigated and the remainder of the universe. The process of the *isolation* of one particular portion of the universe, for purposes of a particular problem, is one of the central concepts in mechanics.

In order to isolate correctly a mechanical system, it is necessary to make a clear distinction between forces which are external to

the system, and those which are internal. An external force is a force exerted on a system or some part of a system by a body outside of the system. An internal force is exerted on one part of the system by some other part of the same system. A particular force, of course, may be at the same time an internal force so far as one system is concerned, and an external force so far as another system is concerned.

Consider the two blocks of weight W_1 and W_2 as shown in Fig. 1–17a. In Fig. 1–17b the two blocks together have been considered as the mechanical system which is to be isolated, and a

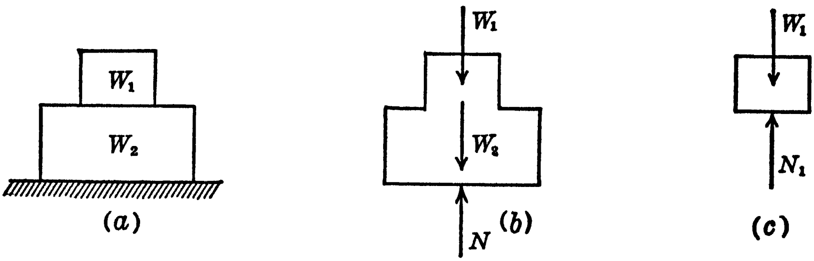


FIG. 1–17

diagram has been drawn showing all of the external forces acting on the system. Such a diagram is called a *free-body diagram. The preparation of a free-body diagram should be the first step in every mechanics problem.* The purpose of the free-body diagram is to indicate that portion of the physical arrangement which is isolated as a mechanical system. The external forces acting on this system are the weights of the blocks W_1 and W_2, and the force N exerted on the system by the supporting plane. The force between the two blocks would in this system be an internal force and hence would not appear in the free-body diagram.

In Fig. 1–17c the upper block alone has been isolated as a mechanical system, and the free-body diagram has been drawn. In this system the external forces acting are the weight of the block W_1 and the force N_1 exerted on the block W_1 by the block W_2. In this case the force N_1 is an external force so far as the system shown in (c) is concerned, but an internal force so far as the system of (b) is concerned.

The system which is selected for isolation need not follow the physical boundaries of the elements concerned, as it does in the

example of Fig. 1–17. The free-body diagram may include only one portion of a body, as in the case shown in Fig. 1–18.

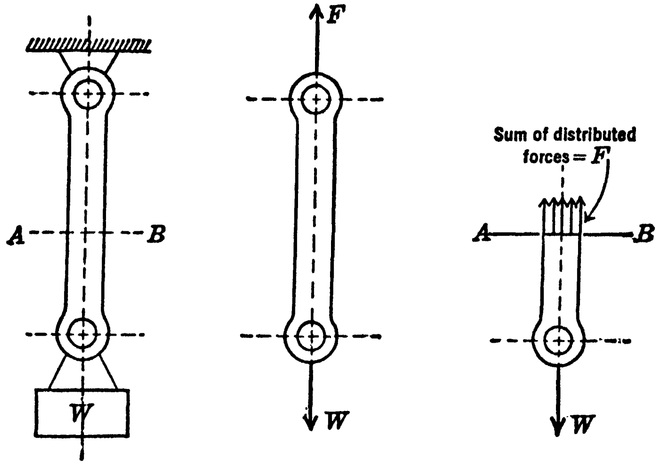


FIG. 1–18

Fig. 1–18b is the free-body diagram of the tension link shown in Fig. 1–18a. The weight of the link has been assumed to be negligible compared to the weight W. In Fig. 1–18c we imagine that the link has been cut at the section A–B, and we draw a free-body diagram of the lower portion of the link. The force that each element of the upper portion exerts on the corresponding element of the lower portion is thus shown as a uniformly distributed system of forces over the upper section of the bar. The internal forces in the bar, which did not appear in Fig. 1–18b, are external forces for the section of the bar shown in Fig. 1–18c. It is apparent that such an imaginary cut must always be made if it is desired to study the internal forces in a body.

The question of precisely what should be included in the free-body diagram cannot be answered in general, and it is at this point that the ingenuity of the investigator must show itself. The unknown forces which are to be determined must appear, however, as external forces in some free-body diagram. For any type of force system there is a maximum number of equations which can be written, and hence a maximum number of unknown forces which can be determined. Frequently it will be found that the free-body diagram which will show the desired forces as external

forces will involve so many unknown forces that no solution will be possible. In this event it may be necessary to begin the problem by considering some other free-body diagram, from which some of the unknowns may be determined.

It is of particular importance that all of the external forces acting upon the system be shown in the free-body diagram. There is sometimes a tendency to assume that certain forces, such as gravity forces, are so obvious that it is not necessary to take the time to put them down on the free-body diagram. Such practices inevitably lead, sooner or later, to the omission of such forces from consideration. Every force, no matter how obvious, should be included in the free-body diagram unless one of the assumptions or conditions of the problem is that the force is equal to zero. Even in such a case it is well to include the force, and then to write the statement that the force is equal to zero, since other conditions may exist under which the force cannot be neglected.

In considering the forces which may act upon a body, it should be noted that any point at which the body makes contact with another body is a point at which a force may exist. It may be possible, from other data given in the problem, to determine the direction of the force at the point. For example, the direction of the force which acts at a frictionless support is normal to the surfaces in contact. A second instance in which the directions can be found easily is that of a system in equilibrium under the action of only two forces. Since two forces can be in equilibrium only if they are equal in magnitude, opposite in direction, and lie along the same line of action, they must have the same direction as the line joining their points of application. This means that the forces acting at the ends of such members as cables and connecting rods have the direction of the cable or connecting rod, providing that the weight of the member is negligible. It must also be kept in mind that forces may exist without physical contact between bodies, as in the case of gravitational, electromagnetic, or electrostatic forces.

EXAMPLE 1.

(a) Two smooth cylinders are shown (Fig. 1–19) resting in a smooth trough.

(b) A free-body diagram of the two cylinders considered as one body is shown in part (b). Since all of the surfaces are fric-

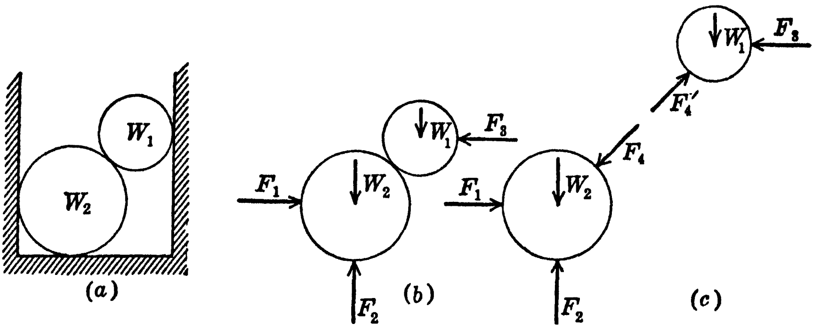


Fig. 1–19

tionless, the forces between surfaces in contact are normal to the surface. Note that the forces between the two cylinders are internal forces, so far as the system consisting of the two cylinders is concerned, so that these forces do not appear on the free-body diagram.

(c) In part (c) are shown the two free-body diagrams for the two cylinders, each considered as a separate mechanical system. In this case, considering either cylinder alone as a free-body diagram, the force exerted by the other cylinder becomes an external force in that system.

Example 2.

(a) A load F is applied to a rigid pinned framework as shown in Fig. 1–20a. The end of the member CE rests on frictionless

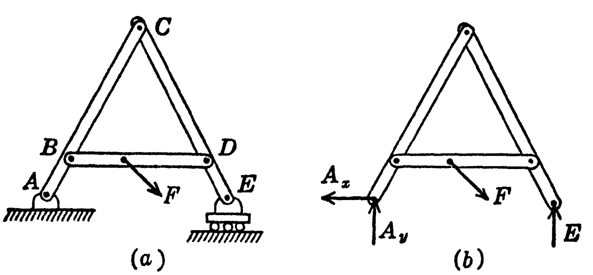


Fig. 1–20

rollers on a horizontal plane. The weights of the members are small compared to the other forces acting and may be neglected.

(b) In (b) is shown a free-body diagram of the whole structure. Since the rollers at E are frictionless, the force exerted on the framework at E must be normal to the surface. At A the pin can transmit a force in any direction, hence the direction as well as the magnitude of the force is unknown. We represent the force at A, therefore, by two unknown rectangular components A_x and A_y.

(c) Free-body diagrams of each of the three members making up the structure are shown in (c). At the pin connections both

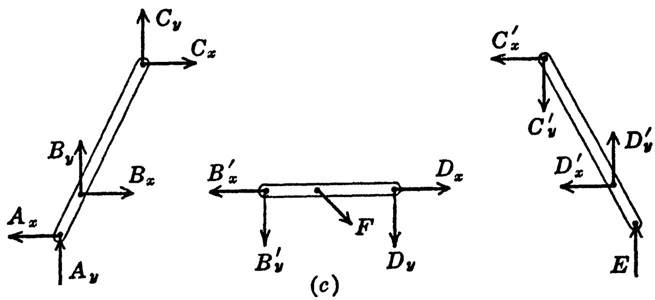


FIG. 1-20

the magnitudes and directions of the forces are unknown, as indicated by the two unknown rectangular components of the force. The forces exerted between two members, which were internal forces in the system shown in (b) above, are now external forces so far as the separate members are concerned. These forces exist as equal and opposite pairs, and hence cancel out in (b).

EXAMPLE 3.

(a) In Fig. 1–21a the elements of a simple band-type brake are shown. The brake drum rotates clockwise, a force P tightens

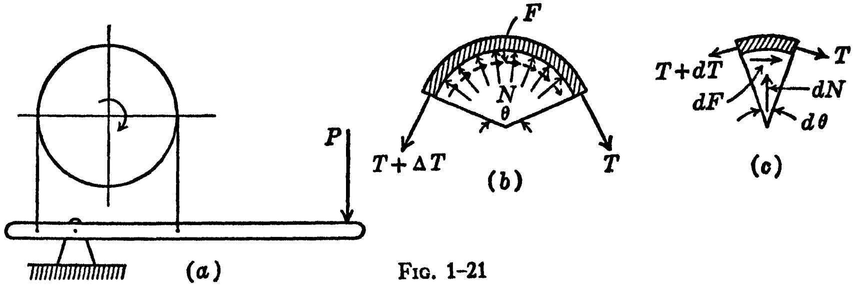


FIG. 1-21

the brake band on the drum and thus provides increased frictional braking torque, forming a so-called self-energizing brake system.

(b) In (b) is shown a free-body diagram of a portion of the brake band subtended by the angle θ. The weight is small compared to the other forces acting and hence is neglected. If it is assumed that the brake band is perfectly flexible, i.e., can support no bending, the forces T and $T + \Delta T$ will be tangent to the drum as shown. The frictional forces between the drum and the band are shown as a system of uniformly distributed forces tangent to the drum, and the reaction force between the drum and band is shown as a uniformly distributed system of normal forces.

(c) In this figure is shown the free-body diagram of a portion of the brake band subtended by an infinitesimal angle $d\theta$. The uniformly distributed frictional forces have been replaced by their resultant dF and the uniformly distributed normal forces have been replaced by their resultant dN.

Example 4.

(a) Fig. 1–22a shows a beam supported at two points and loaded by a vertical force P in such a way as to cause bending in a vertical plane.

(b) In (b) is shown a free-body diagram of that portion of the beam to the right of the section A–B. The vertical force

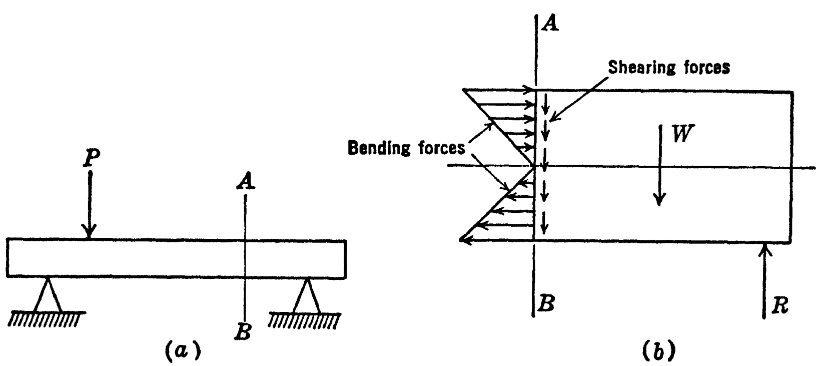


Fig. 1–22

exerted on the beam by the right support is shown as R. The weight W of the portion of the beam to the right of the sec-

tion A–B is shown acting at the center of gravity of the element. The forces which are exerted on this section of the beam by the left section, which we imagine to have been removed, are shown as two systems of forces, a system of horizontal forces due to bending in the beam, and a system of vertical forces due to shearing action. Neither of these two systems of forces is in this case uniformly distributed. As we shall find out later by an analysis of the problem, the bending forces will be in tension at the lower edge of the beam, compression at the upper edge of the beam, and will pass through zero at some intermediate section. The shearing forces will be zero at the top and bottom edges of the beam, and will be a maximum at some intermediate position. All that we do in the present free-body diagram is to indicate the presence of forces having unknown magnitudes and distributions. The particular form of the forces shown in the figure could not be determined without further analysis.

EXAMPLE 5.

(a) Fig. 1–23a shows a section of thick-walled tube subjected to an internal pressure p.

(b) Fig. 1–23b shows the free-body diagram of the very small element of the body which is shaded in (a). All of the six faces

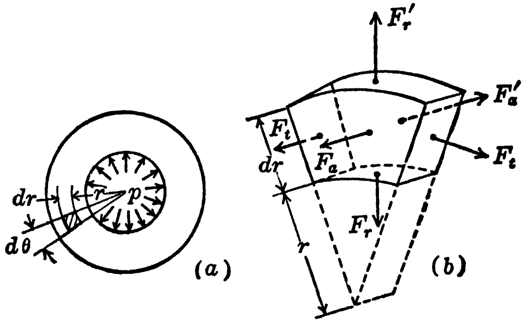


FIG. 1–23

of this element are acted upon by forces which may be considered as uniformly distributed over the infinitesimally small areas of the faces. These uniformly distributed forces have been replaced by their resultant forces, acting at the centroid of the faces of the element. The weight of the element has

been assumed to be very small compared with the other forces acting on the system.

EXAMPLE 6.

Figure 1–24 shows a free-body diagram of a rectangular prism of fluid located at some distance below the surface of the fluid.

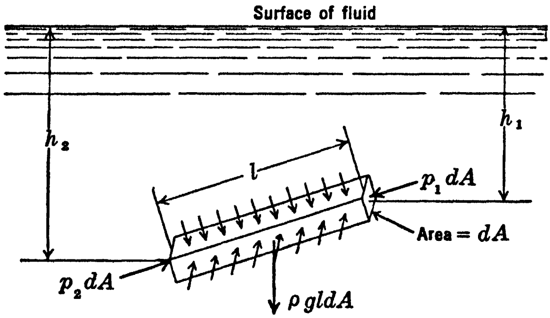


FIG. 1–24

The infinitesimal cross-section area of the prism is dA, and the length of the prism is l. On each face of the prism, the force acting will be equal to the pressure times the area. The gravity force will be the volume of the prism times the specific weight.

EXAMPLE 7.

(a) In Fig. 1–25a are shown the basic elements of the oil drop experiment by means of which Millikan determined the charge

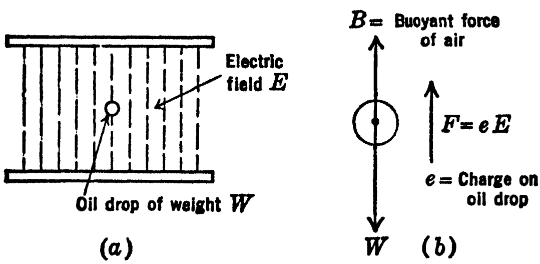


FIG. 1–25

of the electron. The oil drop of weight W is shown in an electric field of intensity E.

(b) A free-body diagram (b) of the oil drop is shown. The drop is in equilibrium under the action of the buoyant force of the

air, the weight, and the force due to the presence of the charge in the electric field.

In all of the examples given above, the forces acting on the system have been balanced, and hence the systems have been in equilibrium. The usefulness of the free-body diagram, however, is not limited to problems in statics only. In dynamics problems, in which the motions which result from unbalanced force systems are studied, it is also of the highest importance to have a clear picture of all of the forces acting upon the system.

Some examples of free-body diagrams as applied to dynamics problems follow:

EXAMPLE 8.

(a) Figure 1–26(a) shows a schematic view of one cylinder of a reciprocating internal combustion engine.

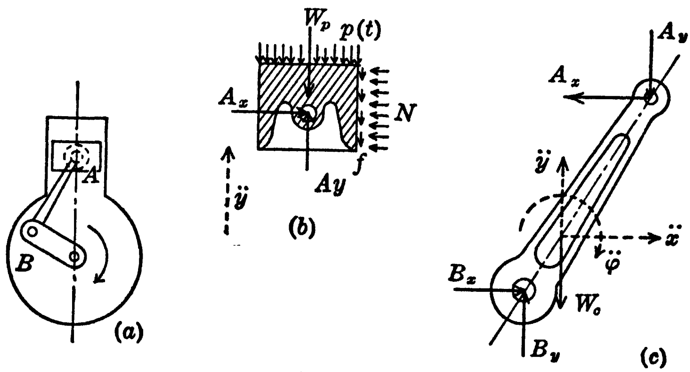


FIG. 1–26

(b) Figure 1–26(b) is a free-body diagram of the piston A of the engine. The forces $p(t)$, which are approximately uniformly distributed over the top of the piston, are the gas-pressure forces resulting from the explosion of the gas-air mixture above the piston. These forces vary with respect to time, hence they are indicated as functions of time. The force exerted by the connecting rod AB on the piston is unknown in both magnitude and direction, hence it is indicated in the diagram as the two rectangular components A_x and A_y. The forces exerted on the piston by the cylinder wall are shown approximately as two uniformly distributed systems of forces,

normal forces N, and frictional forces f. The dotted vector indicates the direction of the acceleration of the piston. The weight of the piston W_P is shown acting at the c.g. of the piston.

(c) In this figure (c) a free-body diagram of the connecting rod AB of the engine is shown. The force exerted on the rod by the piston at A is shown as two rectangular components A_x and A_y. These are, of course, equal and opposite to the forces exerted by the rod on the piston as shown in diagram (b) above. The forces exerted by the crankshaft on the lower end of the rod are shown as two unknown rectangular components B_x and B_y. The weight of the connecting rod W_C is shown acting at the center of gravity of the rod. The dotted vectors indicate that there is an acceleration of the c.g. of magnitude $\ddot{x}$ in the x-direction, of magnitude $\ddot{y}$ in the y-direction, and an angular acceleration of magnitude $\ddot{\varphi}$.

EXAMPLE 9.

Fig. 1–27 shows the free-body diagram of an airplane considered as a rigid body. The longitudinal axis of the airplane makes an

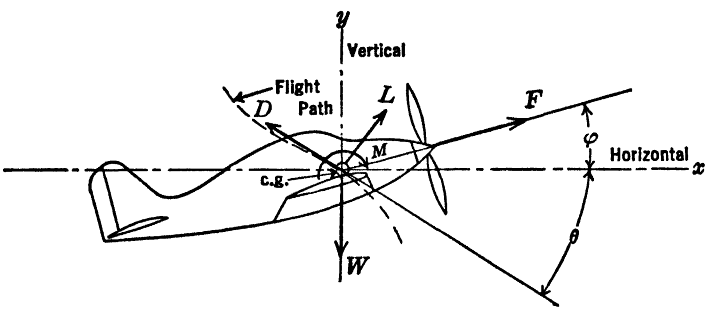
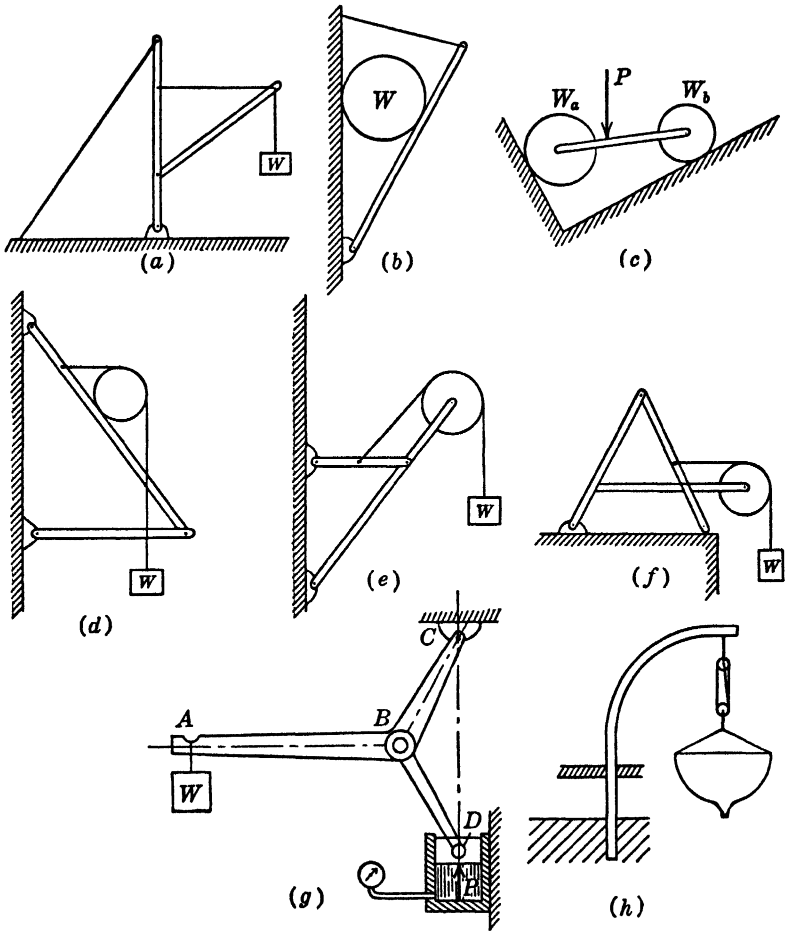


FIG. 1–27

angle φ with the horizontal, and the flight path makes an angle θ with the horizontal as shown. The weight of the airplane W is shown acting at the center of gravity of the airplane. There is a lift force L, perpendicular to the flight path, a drag force D tangent to the flight path, and a moment M which tends to pitch the airplane about an axis perpendicular to the paper. The force F is the propeller thrust.

PROBLEM

48. For each of the following structures, draw a free-body diagram of the complete structure, and for each of the members of the structure.



Prob. 48

CHAPTER II

THE EQUILIBRIUM OF FORCE SYSTEMS

PROP. 6: Commensurable magnitudes balance at distances reciprocally proportional to their weights.

PROP. 7: And now, if the magnitudes be incommensurable, they will likewise balance at distances reciprocally proportional to the magnitudes.

—Archimedes (287–212 B.C.), *On Plane Equilibrium.*

In the preceding chapter the problem of reducing any type of force system to the simplest possible equivalent force system has been solved. In many practical problems the forces acting upon a system are balanced so that the resultant force is equal to zero. In the present chapter the conditions under which this equilibrium condition is obtained will be investigated. We shall first establish the equations of equilibrium for the general three-dimensional force system.

If the general force system is to be in equilibrium, both the resultant force and the resultant couple must be equal to zero, and the conditions for equilibrium become:

$$\boldsymbol{R} = \Sigma \boldsymbol{F} = 0$$
$$\boldsymbol{C} = \Sigma \boldsymbol{M} = 0$$

It is convenient usually to describe the force system in a rectangular coordinate system. Since the force and the couple vector can be zero only if their three rectangular coordinates are zero, the general equations of equilibrium become:

$$\begin{cases} \Sigma F_x = 0 \\ \Sigma F_y = 0 \\ \Sigma F_z = 0 \\ \Sigma M_x = 0 \\ \Sigma M_y = 0 \\ \Sigma M_z = 0 \end{cases}$$

Since it requires three quantities to specify the force vector in any coordinate system, and an additional three to specify the

couple vector, there are six independent equations available to express the conditions of equilibrium. For a system of forces in equilibrium, the sum of the forces along any arbitrary axis is zero, and the sum of the moments about any arbitrary axis is zero, so that any number of valid equations can be written. All these equations, however, will not be independent. For any particular force system, the number of independent equations of equilibrium which can be written is equal to the number of quantities which must be given in order to specify completely the resultant of the force system. A knowledge of the number of independent equations which can be written for any particular system is of importance because this specifies the maximum number of unknown quantities which can be determined by the equations of equilibrium.

The number of such independent equations available for the various force systems are:

(1) *The General Three-dimensional Force System.* Six independent equations may be written. These may be all moment equations, or as many as three of the equations may be force equations.

(2) *The General Concurrent System.* The resultant of this system is a single force which requires three components for its complete specification. There are thus three independent equations of equilibrium. These equations may be force equations, moment equations, or both force and moment equations.

(3) *The General Coplanar System.* Three quantities will specify completely the resultant of a general system of forces in one plane, so three equilibrium equations are available. Either force or moment equations can be used, except that a maximum of two force equations can be written.

(4) *The General Parallel System.* In a system of parallel forces in space three equations can be written. These will ordinarily be one force equation and two moment equations.

(5) *Coplanar Parallel System.* Two equations, either a force and a moment equation, or two moment equations may be written.

(6) *Concurrent Coplanar System.* Two equations, either forces, moments, or moments and forces, may be written.
(7) *Collinear Systems.* One equation, either a force or a moment equation, may be used.

In practice, one selects the type of equation and the axes along which forces are to be summed or about which moments are to be summed, according to the particular conditions of the problem. For most problems the use of certain axes will result in simpler equations than the use of other possible axes, so that one should examine each problem carefully and select the most appropriate set of equations. Moment equations, for example, can often be simplified by selecting a moment axis which intersects the lines of action of several of the forces of the system. In this way equations involving only one or two unknowns can often be obtained.

EXAMPLE. A crane consisting of a vertical rigid bar *AD* which is pivoted at *D* is supported by two guy wires *AE* and *AF* (Fig. 2–1).

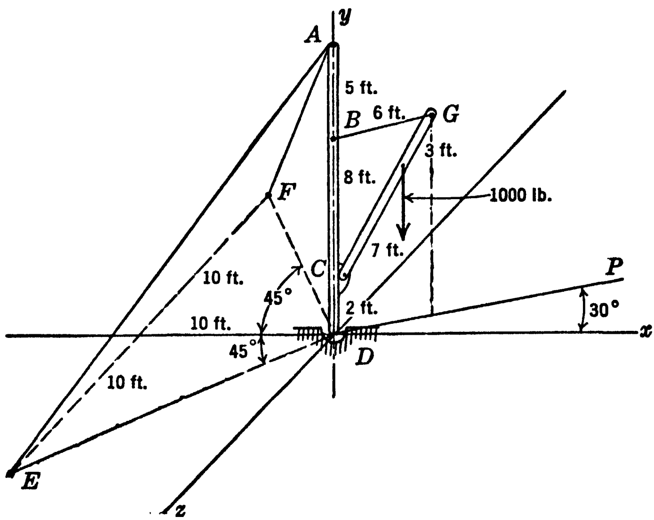


FIG. 2–1

A vertical load of 1000 lb is picked up by the boom *CG* which is supported by the guy wire *BG*, the whole configuration of the system being shown in the diagram. Find all of the forces exerted on the bar *AD*. The weights of the bars are small compared with the other forces acting and hence in this problem may be neglected

Solution. We first draw a free-body diagram (Fig. 2–2) of the bar AD.

The forces in the guy wires at A and B are known to be in the direction of the guy wires, and hence may be drawn in their proper

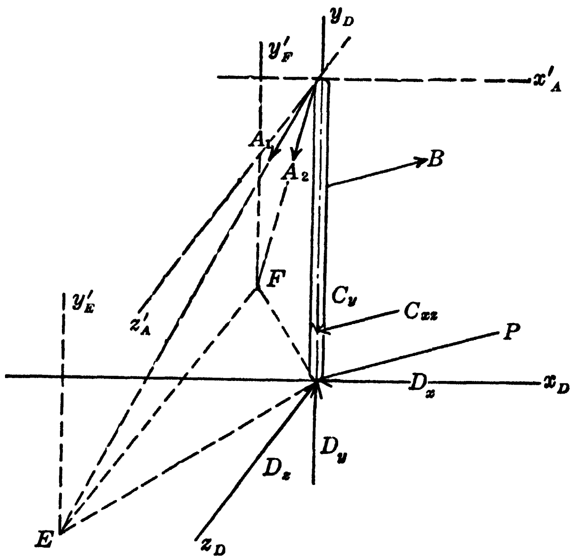


FIG. 2–2

direction on the free-body diagram. The force at C must lie in the plane of the boom CG and the guy wire BG, but since we do not know the direction in this plane, we indicate the force by two unknown components C_y and C_{xz} where C_{xz} is parallel to the xz plane. The force at D is completely unknown and may be in any direction in space, so that we indicate this force by three rectangular components D_x, D_y, and D_z. We now note that we have eight unknown quantities in the problem, whereas we can write down only six independent equations for the general force system. It is thus apparent that we cannot determine all of the forces from the free-body diagram shown in the diagram, and we examine the possibility of determining some of the forces from a free-body diagram of some other portion of the structure. We next draw the free-body diagram (Fig. 2–3) of the boom CG.

For this general coplanar system three equations can be written and hence the forces C_y, C_{xz}, and B can be determined. Having

thus found three of the eight unknown elements of the original free-body diagram, the six equations which can be written for that free-body diagram will suffice to complete the solution of the problem.

For the free-body diagram of the boom BC, we have:

$$\Sigma F_P = 0 = C_{xz} - B$$
$$\Sigma F_y = 0 = C_y - 1000; \quad C_y = 1000 \text{ lb}$$
$$\Sigma M_C = 0 = (8)(B) - (\tfrac{3}{5})(7)(1000); \quad B = 525 \text{ lb (tension)}$$

and from the first equation

$$C_{xz} = 525 \text{ lb}$$

Since all of the forces came out with positive signs, we know that we chose the correct directions for the unknown components. Note that the forces B, C_y and C_{xz} in the second free-body diagram are equal and opposite to the forces B, C_y, and C_{xz} in the first free-body diagram.

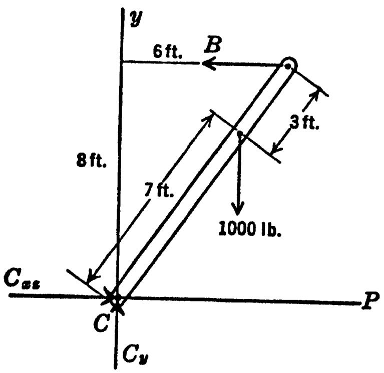


FIG. 2-3

Returning now to the first free-body diagram and writing the equations in each case for the axis which will yield the simplest result, we have:

$$\Sigma M_{EF} = 0 = -10B_x + 2C_x - 10C_y + 10D_y$$
$$= -(10)(525)(0.866) + (2)(525)(0.866) - (10)(1000) + 10D_y$$
$$\therefore D_y = 1364 \text{ lb}$$

$$\Sigma M_{y'A} = 0 = 5B_x - 13C_x - 15D_x$$
$$= (5)(525)(0.866) - (13)(525)(0.866) - 15D_x$$
$$\therefore D_x = -242 \text{ lb}$$

Since D_x comes out with a negative sign, we know that the direction was incorrectly shown on the free-body diagram, and that the force is actually opposite to the force shown there.

$$\Sigma M_{x'A} = 0 = 5B_z - 13C_z + 15D_z$$
$$= (5)(525)(0.500) - (13)(525)(0.500) + 15D_z$$
$$\therefore D_z = 140 \text{ lb}$$

$$\Sigma M_{y'E} = 0 = \frac{10}{\sqrt{(10)^2 + (10)^2 + (15)^2}}(A_2)(20) + 10D_x + 10D_z$$
$$= \frac{20}{20.6}(A_2) - 242 + 140$$
$$\therefore A_2 = 105 \text{ lb}$$

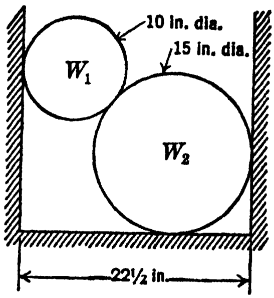
$$\Sigma M_{y'F} = 0 = -\frac{10}{\sqrt{(10)^2 + (10)^2 + (15)^2}}(A_1)(20) + 10D_x + 10D_z$$
$$= -\frac{20}{20.6}A_1 + 242 + 140$$
$$\therefore A_1 = 394 \text{ lb}$$

As a check equation, we may write:

$$\Sigma F_y = 0 = -\left(\frac{15}{20.6}\right)(394) - \left(\frac{15}{20.6}\right)(105) - 1000 + 1364$$
$$= -287 - 77 - 1000 + 1364 = 0$$

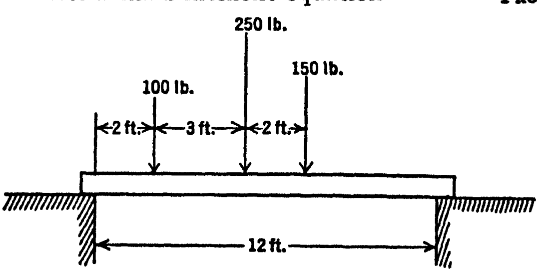
PROBLEMS

49. Two homogeneous circular cylinders are supported in a trough as shown in the figure. The weight of W_1 is 250 lb, and the weight of W_2 is 750 lb. Find the forces exerted by W_1 on W_2, and by the two cylinders on the walls and floor of the trough.



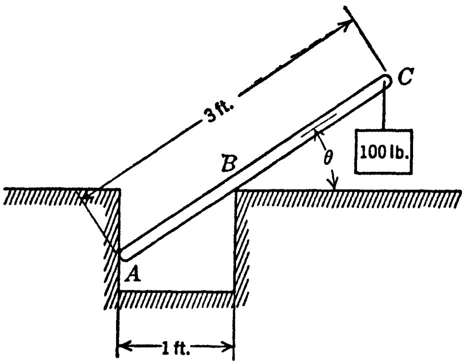
PROB. 49

50. Three loads act on a simply supported beam as shown in the diagram. Find the forces exerted on the beam by the two supports. Solve for these forces from two moment equations, and check by means of a force equation. The weight of the beam is assumed to be small compared to the loads acting and is neglected. Write a third moment equation



PROB. 50

about some point not used for the first two moment equations, and show that this third equation is not independent, but can be formed from the first two equations.



PROB. 51

51. A 100-lb weight is suspended from the end of a 3-ft long weightless rod as is shown in the diagram. Find the angle θ for equilibrium of the system. Assume that the surfaces in contact are frictionless, i.e., the force at B is normal to the rod AC.

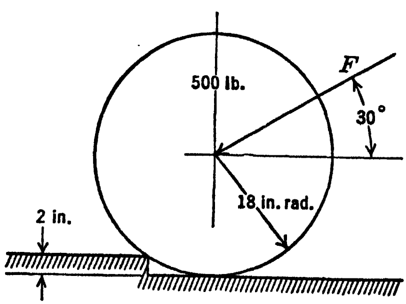
52. Show that for a general coplanar system three moment equations are sufficient to insure equilibrium only in the case that the three moment centers are not colinear.

53. Show that three concurrent forces which are not coplanar cannot be in equilibrium.

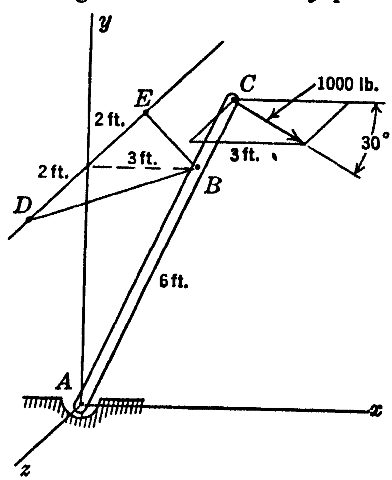
54. A roller has a radius of 18 in. and a weight of 500 lb. What force must be applied as shown in the diagram to just lift the roller over a 2-in. obstacle; i.e., to reduce the vertical reaction on the roller to zero?

55. The pole AC, lying in the xy plane, is supported by two horizontal cables BD and BE as shown in the figure. A 1000-lb horizontal force acts at the end of the pole and makes an angle of 30° with the xy plane as shown. Find the force at A, and the forces in the cables.

56. Show that if three coplanar, non-parallel forces are in equilibrium, they must be concurrent.



PROB. 54



PROB. 55

Three Forces in Equilibrium. As has been shown above in Problem 56, three non-parallel forces in a plane can be in equilibrium only if they pass through the same point. In some cases where it can be seen that forces act upon a structure at only three points, this fact can be used to introduce a simplification into the solution of the problem. The application of this principle will be seen in the following example.

EXAMPLE. A semi-circular arch is pinned at the middle and at the ends as shown in Fig. 2–4a and is loaded on one side by a

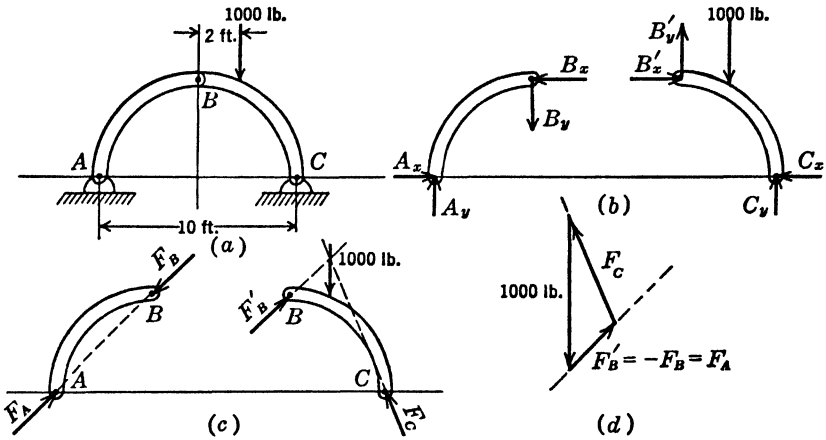


FIG. 2–4

1000-lb force as shown. Find the forces at A, B, and C. Neglect the weights of the members.

Solution. If the free-body diagram of the two parts of the structure (b) or of the structure as a whole is drawn, we find that there are apparently more unknowns than there are equations. We note, however, in (c), that the left-hand member is a two-force member, and hence the forces at A and B must have the direction AB. We also see that the right-hand member is subjected to three forces which must be concurrent. The vector triangle (d) can thus be constructed directly, and the magnitudes of the forces can be determined.

Complete Constraint of a Rigid Body in Space. We now consider the total number of constraints or supporting elements which are required to constrain completely a rigid body in space.

Consider the body shown in Fig. 2–5. In Fig. 2–5a one point of the body, O, has been fixed in space by means of the three rigid

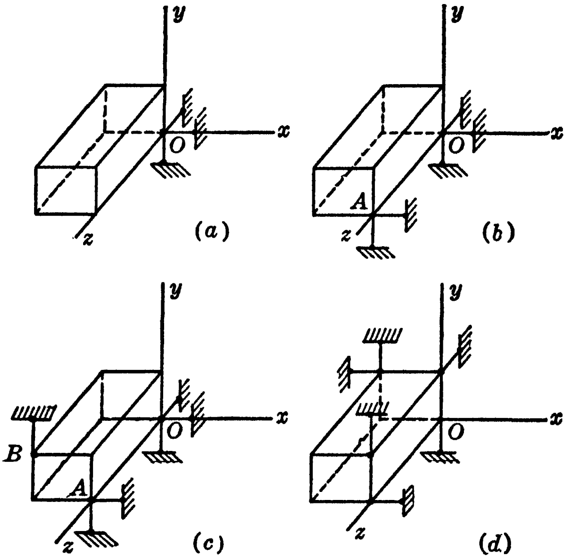


FIG. 2–5

hinged bars as shown. Three non-coplanar bars are required, otherwise motion of the point normal to the plane of the bars can occur. With the constraint shown in Fig. 2–5a any point in the body is free to move in a spherical surface with O as a center. In Fig. 2–5b another point A has been constrained. This can be done by two bars whose plane is normal to one of the bars at O, as shown. The only possible remaining motion of the body is a rotation about the axis OA, which can be prevented by means of a hinged bar which does not intersect the axis OA. This could be done, for example, by the bar shown at B in Fig. 2–5c. It will thus be seen that six elements will constrain completely the body in space. It is not necessary to arrange the elements exactly as shown in Fig. 2–5c. In Fig. 2–5d, for example, another arrangement of the six bars is shown which accomplishes the same purpose. It will be found, however, that complete constraint cannot be obtained with less than six elements for the general space system. It will be seen that this is generally true, since for equilibrium, the six components of the known forces (three force components and three

couple components) must be balanced by six equal and opposite components of the constraining forces. It is not true, however, that any six elements are sufficient to insure constraint, since the relationships between the supports may be such that motion could occur. If all of the bars, for example, were parallel or were arranged in parallel planes, movement could occur normal to the plane. Similarly, if the axes of the six bars could be intersected by one straight line, there would be a possibility of rotation about that line.

From considerations such as those given above, it can be shown that the necessary and sufficient condition for the complete constraint of a rigid body in space is that it be supported by six hinged bars, or some equivalent structure, whose lines of action cannot be intersected by a straight line. This condition includes the case of the parallel bars, whose lines of action may be said to intersect the same straight line at infinity.

If other bars should be used to support the body, in addition to those required for complete constraint, such additional supports are called *redundant* constraints. Since six statical equations can be written for the general force system, it will be apparent that the constraining forces for a body loaded in any manner can always be determined from the equations of statics, providing that no redundant constraints exist. This situation, in which there are no redundant constraints, is termed a statically determinate constraint of the body, while the addition of redundant constraints makes the problem statically indeterminate.

The same conclusions will be found to apply to all of the simpler force systems. In each case the number of supports required for complete constraint is equal to the number of statics equations which can be written for the system.

Degrees of Freedom. In many problems occurring in the design of instruments and machinery, one does not want to constrain a body completely, but wishes to permit certain types of motion or freedom. By the number of degrees of freedom of a system is meant the number of independent coordinates which would have to be specified in order to fix completely the location of every part of the system. A rigid body which could move only along one straight line, for example, would have one degree of freedom, since

one coordinate only, the distance from some arbitrary point to the body, would suffice to fix the position of the body. A body which could only rotate about an axis would also be a one degree of freedom system, since only the angle of rotation is required to fix the location of all parts of the body. Referring to Fig. 2–5a, it will be seen that this system, which is supported by three constraints, has three degrees of freedom, since, for example, three angles about the three coordinate axes would have to be given in order to specify the location of the body. In Fig. 2–5b the addition of two more constraints reduces the number of degrees of freedom to one, since now only rotation about an axis is possible. It will be found in all cases that the number of degrees of freedom of the system is equal to six minus the number of constraints. This principle is useful in design, since it indicates the number of contacts between the body and its supporting structure which are required for the desired type of motion.*

Statically Indeterminate Systems. If there should be more unknowns in a particular free-body diagram, than there are equations, the following possibilities exist:

(1) It may be possible to solve for some of the unknowns from other free-body diagrams. This is illustrated in the example of page 42 and is the first possibility to investigate when too many unknowns are encountered.

(2) Other data may be given, or obtainable, from which some of the unknowns may be determined. For example, a knowledge of the coefficient of friction of surfaces in contact may give an additional equation. Such friction problems are treated in a later chapter.

(3) The system may be constrained in such a way that the statical conditions of equilibrium alone are not sufficient to determine the forces in the system. Such a system is called a *statically indeterminate* system, and can be solved only by bringing into the problem additional information as to the properties of the material of which the members are made, and as to the deformation of the members under load.

* Strong, J., *Procedures in Experimental Physics*, Prentice-Hall, Inc., New York, 1943, p. 585.

In Fig. 2–6 two examples of statically indeterminate systems are shown.

In Fig. 2–6a the systems shown are statically determinate, and the forces can be calculated by the methods already discussed.

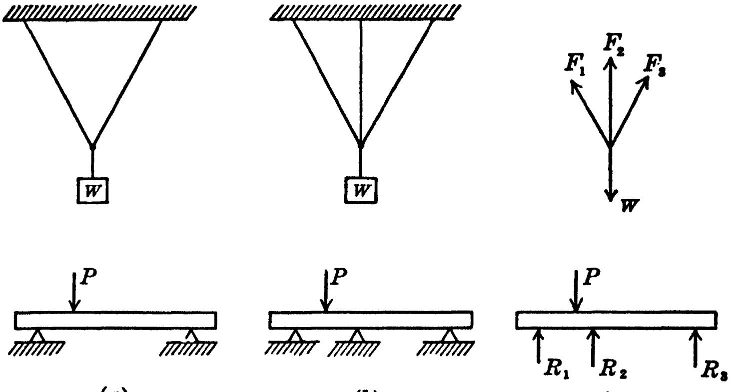
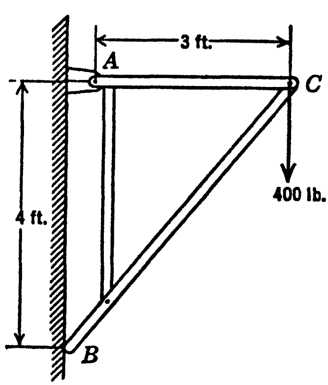


FIG. 2–6

In Fig. 2–6b an additional constraint has been added in each case. In the top figure the weight is supported by three wires instead of two, and in the lower figure the beam is supported at three points instead of two. In these cases there are three unknowns, and only two equations, so that it will not be possible to solve for the forces. Some methods which can be used to solve statically indeterminate problems will be discussed in a later chapter. The solution of such problems is usually considered in connection with the study of the theory of elasticity.

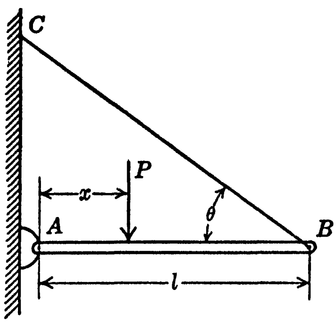
PROBLEMS

57. A rigid framework is supported by a pin at *A* and rests against a frictionless vertical wall at *B*. A vertical load of 400 lb acts at *C*, as shown in the diagram. Find the forces at *A* and *B*. Solve this problem first by making use of the fact that three forces act upon the system, and by drawing a vector triangle. Check the solution by writing the regular equations of equilibrium.



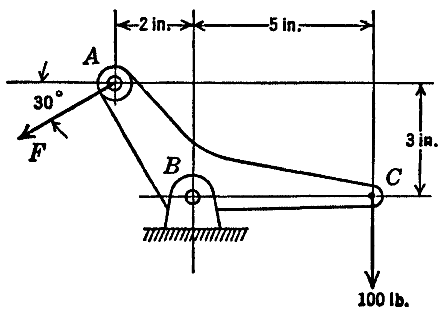
PROB. 57

58. The horizontal member of a crane AB is pinned to a vertical wall at A and is supported at B by the cable BC which makes an angle θ with AB. A load P can be placed at any point along the length AB. Find the force in the cable BC as a function of the position of P and the angle θ. Assuming that the cost of the cable is proportional to the length multiplied by the force, what should be the angle θ for minimum cost of the cable? Assume that the weight of AB is negligible compared to the load P.



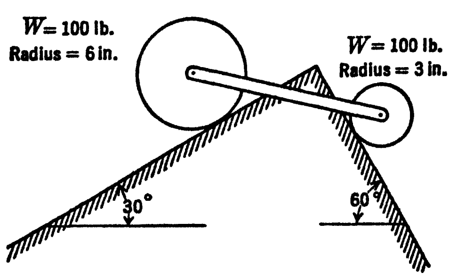
PROB. 58

59. What force F should be applied to the bell-crank at A as shown in the diagram to produce a vertical force of 100 lb at C? What is the force at B under these conditions?



PROB. 59

60. Two cylinders having the dimensions shown in the diagram are joined by a stiff weightless rod and rest on two smooth planes. What is the angle between the connecting rod and the horizontal at the equilibrium position?

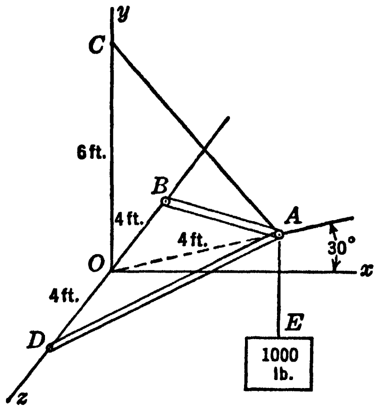


PROB. 60

61. Two stiff weightless bars of equal length AB and AD are pinned at B and D and are supported at A by the cable AC. A 1000-lb vertical load is applied at A. The cable AE lies in the xy plane. Find the force in the cable and in the bars.

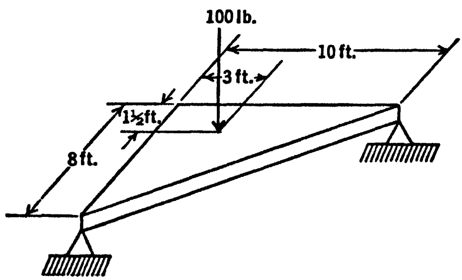
62. A triangular table top weighing 50 lb is supported by three legs and is loaded by a 100-lb vertical force as shown in the diagram. Find the forces in the three legs.

63. A stiff, weightless bar, ABC, is supported at B by two stiff weightless bars, BD and BE. The bars are pinned at A, B, D, and E. At C, loads are applied parallel to the coordinate axes as shown. Find the force at A, and the forces in the members BD and BE.



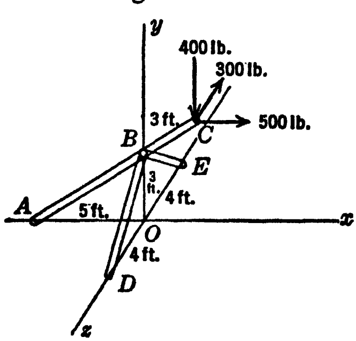
PROB. 61

64. The figure shows a worm and gear combination. The force exerted on the worm by the gear can be resolved into three rectangular components, as shown, an axial component F_a, a tangential component F_t, and a transverse component F_v. For a typical worm gear installa-



PROB. 62

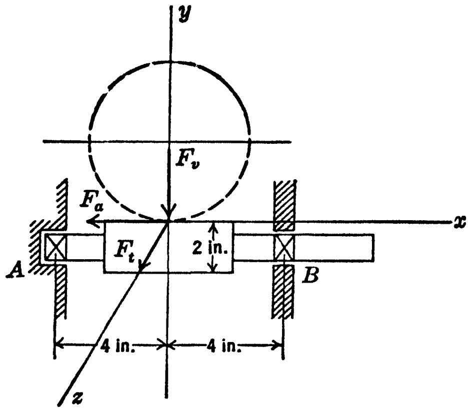
tion, these forces have the values $F_a = 300$ lb, $F_t = 70$ lb, $F_v = 125$ lb. The worm gear is centered between the two bearings, which are 8 in. apart. The pitch diameter of the worm is 2 in., so that the forces F_a, F_t act at a distance of 1 in. from the axis of the worm, as shown in the diagram. Find the forces acting on the bearings, assuming that all of the axial thrust is taken by the left-hand bearing.



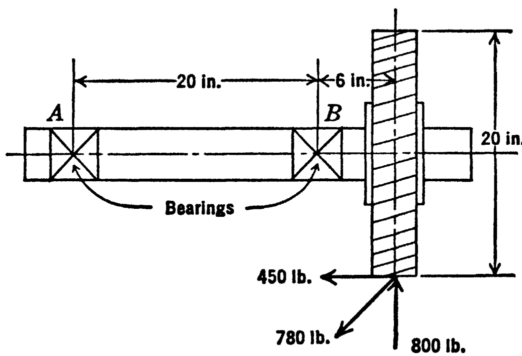
PROB. 63

65. A shaft and a helical gear from a speed reduction unit are shown in the diagram. The forces exerted on the gear teeth are resolved into three rectangular components as shown. As-

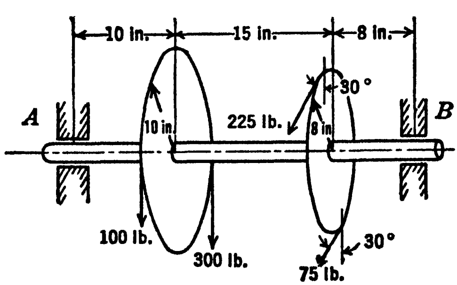
suming that the axial thrust force is all taken by bearing *A*, find the forces acting on the bearings *A* and *B*.



PROB. 64



PROB. 65

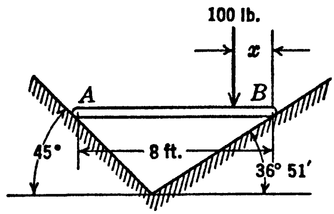


PROB. 66

66. Two pulleys are mounted on a horizontal shaft as shown in the figure. The arc of contact of each pulley is 180°. The belt running over

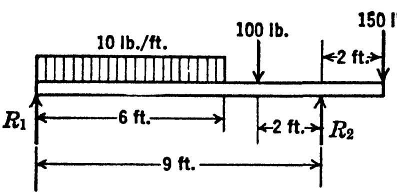
the left-hand pulley is vertical, while the belt running on the right-hand pulley makes an angle of 30° with the vertical. Find the forces on the bearings at A and B.

67. A straight uniform bar weighing 60 lb rests in a horizontal position against two frictionless slopes, as shown in the figure. A concentrated vertical load of 100 lb acts at a distance x from one end of the bar as shown. Find the distance x for equilibrium, and determine the reactions at A and B.

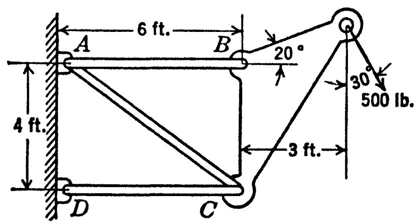


Prob. 67

68. A straight uniform beam of negligible weight is supported in a horizontal position by two vertical reactions R_1 and R_2. The beam is loaded by a uniformly distributed load and two concentrated loads as shown in the figure. Find the magnitudes of the two reaction forces.

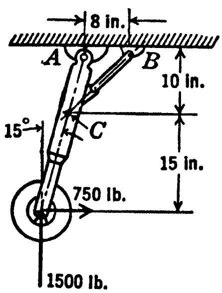


Prob. 68



Prob. 69

69. Find the forces in the bars AB, AC, and CD, of the structure shown in the accompanying figure.



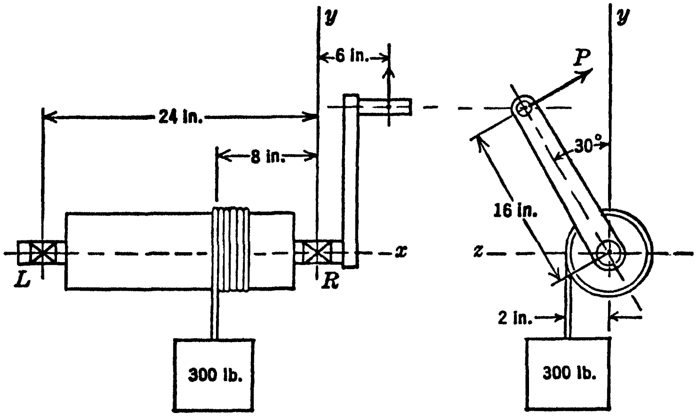
Prob. 70

70. The figure shows diagrammatically the nose wheel of an airplane. Find the force in the compression strut BC, and the components of the force at A.

71. A 300-lb weight is suspended by a cable from the winding drum shown in the figure. What force P must be applied to hold the weight in equilibrium, assuming frictionless bearings? Find the y and z components of the bearing reactions at L and R.

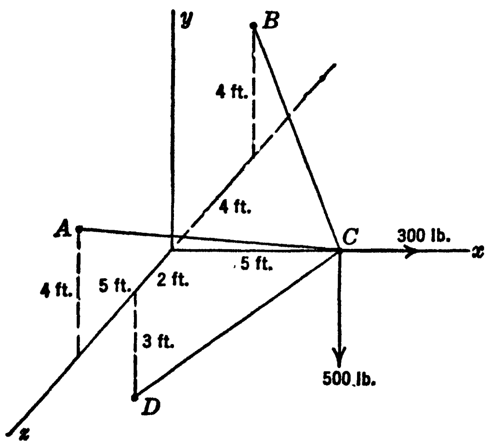
72. A horizontal load of 300 lb and a vertical load of 500 lb are supported by an arrangement of three bars, AC, BC, and DC, as shown in the figure. Determine the forces acting in these bars, assuming that their weight is negligible.

73. A bridge consists of two sections pinned together by smooth pins at C, and to the ground at A and B. A load of 10,000 lb is applied



PROB. 71

to the bridge as shown, through a structure whose weight will be neglected. The weight of each portion of the bridge is 2000 lb, and this

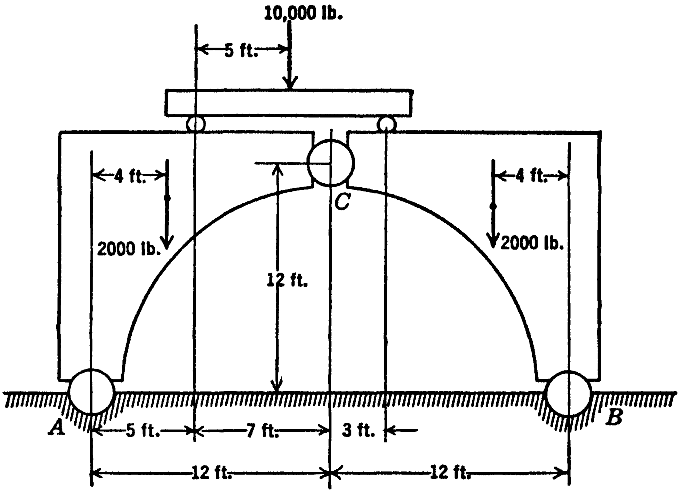


PROB. 72

gravity force acts at a point 4 ft from the supporting pins. Find the forces acting on the pins at A, B, and C.

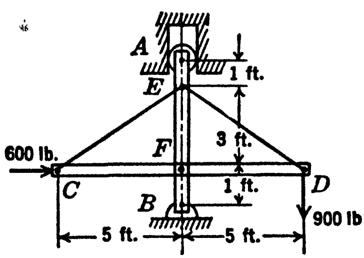
74. In the structure shown in the figure, the members CE and DE are cables, and the force between the support and the frictionless wheel at

A is horizontal. Find the force at *A* and *B*, the force exerted by the pin at *F* upon the vertical member *AB*, and the forces in the cables. Neglect

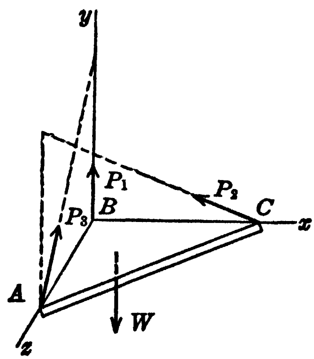


Prob. 73

the weights of the members. If the bars *CE* and *DE* could take compression as well as tension, would the solution to the problem be altered?



Prob. 74

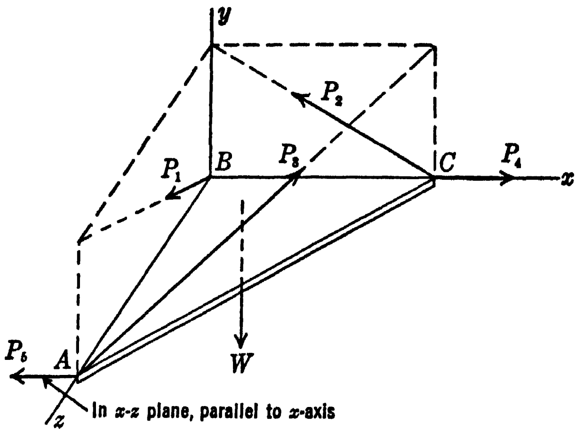


Prob. 75

75. A triangular plate of weight *W* is supported in a horizontal position by three bars. The forces in the bars are P_1, P_2, and P_3 as shown in the diagram. Add the minimum number of supports, at *A*, *B*, or *C*, which will maintain the system in equilibrium. With these additional bars will the plate withstand loading in all directions?

76. Is the horizontal plate shown in the figure completely constrained by the bars P_1, P_2, P_3, P_4, and P_5? If the plate is not completely

constrained, what additional bars should be added at *A*, *B*, or *C* to complete the constraint?



PROB. 76

CHAPTER III

THE PRINCIPLE OF VIRTUAL DISPLACEMENTS

In general, I believe that I may assert that all the general principles which may still be discovered in the science of equilibrium will be only this same principle of virtual velocities, looked at from different points of view, and differing from it only in the form of statement.—Lagrange, *Mécanique Analytique* (1788).

In the development of the principles of statics outlined in the preceding pages, we have adopted only one of several possible points of view concerning the basic elements of the subject. In considering such a problem as the equilibrium of two weights, Fig. 3–1a, we have emphasized the fact that the significant quantities involved are the magnitudes W_1 and W_2 of the forces and

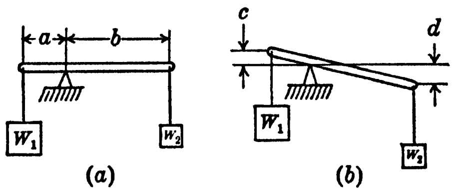


Fig. 3–1

the distances a and b of the forces from the pivot point. The development of these ideas, starting with Archimedes, and brought to their modern form by Varignon, finally culminated in the concept of the moment of a force about a point. In the problem of Fig. 3–1, this concept of a moment would lead to the following equilibrium equation: $(W_1)(a) = (W_2)(b)$. It was perceived, however, by Galileo and Stevinus, that it would be just as logical to focus the attention upon another significant quantity. Suppose (Fig. 3–1b) that we imagine that the lever and weight system has tipped through some small angle, thus raising W_1 through a distance c and lowering W_2 through a distance d. From similar triangles we can say that $a/c = b/d$; substituting this in the above equilibrium equation, we find $(W_1)(c) = (W_2)(d)$. We may now consider as the significant quantities in the problem, the forces W_1

and W_2 and the distances c and d, through which the weights have moved.

This second form of the equation is equivalent to the previous statement, so far as any practical solution of the problem is concerned. We have, however, approached the problem in a somewhat different way. In the first case we balance the moments of the system about the pivot point; in the second case we examine the work which is done by the forces of the system during some possible motion of the system. The universal applicability of this second point of view to the problems of mechanics was first indicated in 1717 by John Bernoulli, and this Principle of Virtual Displacements was taken as the foundation for the science of mechanics by Lagrange in his *Mécanique Analytique*.

Since the principle of virtual displacements introduces no new information into the science of mechanics, we cannot, by its use, solve problems which we would be unable to solve by the methods previously discussed. For certain types of problems, however, the principle leads directly to particularly simple results, and it is for this reason, and for the additional reason that the principle is the starting point for several important developments in dynamics, that we shall here consider the principle of virtual displacements.

The Principle of Virtual Displacements for a Free Particle. Consider a particle which is free to move in any direction, and which is held in equilibrium by some general system of forces whose resultant is zero. In Fig. 3–2 one of the forces of the system, $\boldsymbol{F}$, is shown, along with its three rectangular components.

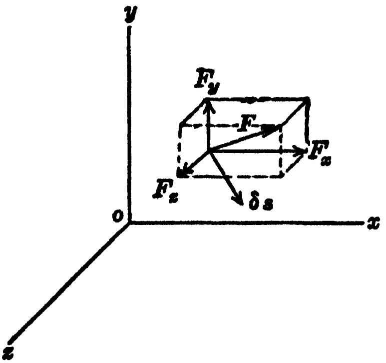


Fig. 3–2

We now imagine that the particle is given some arbitrary infinitesimal displacement $\delta \boldsymbol{s}$. This displacement is arbitrary in the sense that it can be in any direction.

We next compute the work δW which is done by the force $\boldsymbol{F}$ during the displacement $\delta \boldsymbol{s}$. This work δW is the scalar product of $\boldsymbol{F}$ and $\delta \boldsymbol{s}$.

$$\delta W = \boldsymbol{F} \cdot \delta \boldsymbol{s}$$

For all of the forces in the system:

$$\delta W = \Sigma \boldsymbol{F} \cdot \delta \boldsymbol{s}$$

Writing this in terms of components along a system of rectangular coordinates, we have:

$$\delta W = \Sigma(F_x \boldsymbol{i} + F_y \boldsymbol{j} + F_z \boldsymbol{k}) \cdot (\delta x \boldsymbol{i} + \delta y \boldsymbol{j} + \delta z \boldsymbol{k})$$
$$\delta W = \Sigma(F_x \, \delta x + F_y \, \delta y + F_z \, \delta z) = R_x \, \delta x + R_y \, \delta y + R_z \, \delta z$$

where R_x, R_y, and R_z are the rectangular components of the resultant of the system.

If we consider the case in which the forces acting on the particle form a system in equilibrium, we have: $R_x = 0$; $R_y = 0$; $R_z = 0$. Thus $\delta W = 0$, for any values of δx, δy, δz.

We next examine the conditions under which the converse statement is true, i.e., the conditions under which the statement $\delta W = 0$ is both necessary and sufficient for the equilibrium of the system. It will be seen that the only case in which $\delta W = 0$ would not be sufficient to insure equilibrium would be that in which one of the displacement components would be zero. If, for example,

$$\delta x \neq 0; \quad \delta y \neq 0; \quad \delta z = 0$$

then δW could be zero in the presence of a force component F_z. If, however, in our definition of the displacement $\delta \boldsymbol{s}$ we require that δx, δy, and δz be all different from zero, which we imply by the word "arbitrary," then the expression $\delta W = 0$ becomes both necessary and sufficient for the equilibrium of a particle.

We therefore define a *virtual displacement* of a free particle as any arbitrary, infinitesimal displacement for which δx, δy, and δz are different from zero. We use the notation $\delta \boldsymbol{s}$ instead of $d\boldsymbol{s}$ to indicate that the displacement is arbitrary, and can be taken in any direction.

The principle of virtual displacements * thus states that the condition for the equilibrium of a free particle is that the work

* This principle of virtual displacements is in some books called the principle of "virtual velocities." The application of the method would be the same, of course, whether one conceives of giving the various points of the system certain velocities, or certain displacements, since the ratios between the velocities of the various points would be the same as the ratios between the displacements of these points.

done upon any virtual displacement of the particle should be equal to zero.

$$\delta W = \Sigma(F_x \, \delta x + F_y \, \delta y + F_z \, \delta z) = 0$$

The quantity δW is called the virtual work of the system, and the above principle is often called "the principle of virtual work."

The Principle of Virtual Displacements for a Constrained Particle. In many cases the particle under consideration will not be completely free, but will be constrained by various connections to move in a certain way. For example, the particle might be a bead sliding along a wire, in which case the geometrical constraint would be that the particle is required to follow the wire. In another case the particle might be required to remain on a particular surface, as in Fig. 3–3a. In all such cases the behavior of the constraint is to introduce a *reactive force* into the problem which enforces the constraint.

In Fig. 3–3b is shown the free-body diagram of the particle. The active forces in this case have been assumed to be the weight of the particle W, and the force F in the string which holds the particle in equilibrium, and the reactive force due to the constraint is the force N, normal to the surface in the case of a frictionless contact.

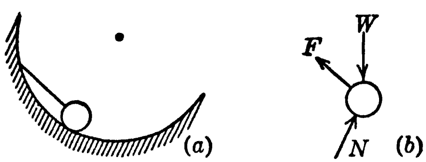


FIG. 3–3

Such a normal constraint, corresponding to frictionless surfaces in contact, is called an *ideal constraint*, and we shall limit our discussion, in the following pages, to such ideal constraints.

In the case of a constrained particle, it will be necessary to alter somewhat our definition of a virtual displacement, since it is clear that certain displacements, such as those normal to a constraining surface, will not be possible. In such a case the virtual displacement cannot be any arbitrary displacement, but must be restricted to those displacements which are permitted by the constraints. We thus define a virtual displacement as an infinitesimal displacement of the particle which is *compatible with the constraints.* The word "virtual" therefore acquires the significance of the word "possible," in that it represents all of those infinitesimal displacements which are consistent with the geometry of the sys-

tem. It will be seen that this more general definition of a virtual displacement includes the definition given for a free particle as a special case.

The equation expressing the principle of virtual displacements can now be written as follows, where we have separated the forces into two types, the active forces $\boldsymbol{F}$ and the ideal reactive forces $\boldsymbol{F}_r$:

$$\Sigma[(F_x + F_{xr})\,\delta x + (F_y + F_{yr})\,\delta y + (F_z + F_{zr})\,\delta z] = 0$$

This equation will still represent the conditions of equilibrium under our new definition of a virtual displacement. In the case of a constraining surface, for example, such virtual displacements would lie in a plane tangent to the surface. Our equation would then prohibit any resultant force in that plane, while the constraint itself insures that the forces normal to the plane balance out.

We next write the above equation in the following form:

$$\Sigma(F_x\,\delta x + F_y\,\delta y + F_z\,\delta z) + \Sigma(F_{xr}\,\delta x + F_{yr}\,\delta y + F_{zr}\,\delta z) = 0$$

and we note that the group of terms containing the reactive forces F_{xr}, etc., will drop out, since the forces will in all cases be normal to the displacements, and will hence do no work. This will always be true, since the ideal reaction is normal to the constraining surface, and the virtual displacement is parallel to the constraining surface. Our equation thus becomes:

$$\Sigma(F_x\,\delta x + F_y\,\delta y + F_z\,\delta z) = 0$$

where F_x, F_y, and F_z are the active forces of the system.

We may thus state the principle of virtual displacements for a constrained particle as: *The work done by the active forces of the system during any arbitrary displacement of the system which is compatible with the constraints is equal to zero.*

The Principle of Virtual Displacements for a System of Particles. The principle of virtual displacements can be extended directly to include rigid bodies, which may be thought of as systems of particles. There will be forces between the various particles forming the body, of such a nature that the distances between the particles will remain fixed. These internal forces, however, will always occur in equal and opposite pairs, so that the total

work done by them during any motion of the whole body as a rigid body will be equal to zero.

Consider the rigid framework shown in Fig. 3–4a, supported by a frictionless pin at O, and acted upon by the three external forces F_1, F_2, and F_3, which maintain the system in equilibrium. In

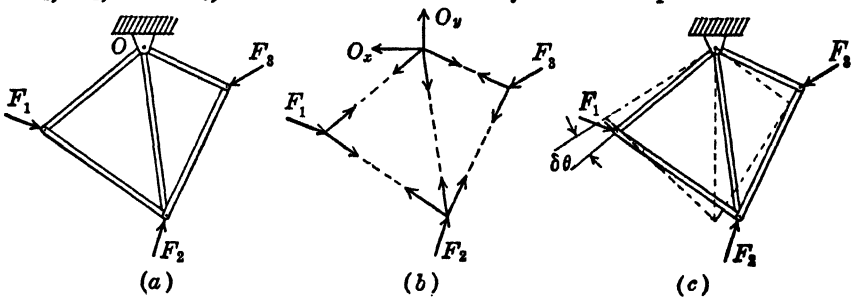


FIG. 3–4

Fig. 3–4b, all the forces which are involved in the system are shown. They will be seen to be of three kinds: first, the active external forces F_1, F_2, and F_3; second, the internal forces in the rigid bars, which occur in equal and opposite pairs; and third, the reaction force at the pin O, which is shown as two rectangular components. If now, we imagine the virtual displacement of the system, as shown in Fig. 3–4c, in which the whole framework has been rotated through some small angle about O, we will see that the forces of the second and third kind will do zero work whether or not the system is in equilibrium, so that for the equilibrium condition we need only consider the first kind of force, and write

$$\Sigma(F_x\,\delta x + F_y\,\delta y + F_z\,\delta z) = 0$$

where F_x, F_y, and F_z are the components of the active forces only, and the summation sign indicates that all of such active forces are to be included. We can thus state in general for any system consisting of frictionless constraints and rigid bars that the necessary and sufficient condition for the equilibrium of the system is that the total work done by all of the active forces of the system, upon any arbitrary infinitesimal displacement of the system compatible with the constraints, is equal to zero. This is the most general statement of the principle of virtual displacements.

From the examples given above, and those which will follow, it will be seen that the particular advantage of the principle of

virtual displacements is realized in those problems in which one is interested only in the equilibrium relationship between the active forces of the system and is not interested in computing the values of reactions or of internal forces. This does not mean that the principle of virtual displacements can never be used to determine reactions. It is often possible to select the force system to be determined in such a way that the reactive forces can be treated as active forces and can, therefore, be treated by the methods outlined above.

EXAMPLE 1. A rigid, weightless bar rests against two frictionless walls as shown in Fig. 3–5. It is held in equilibrium by two forces F_1 and F_2 parallel to the wall. What is the relationship between F_1 and F_2 in the position shown?

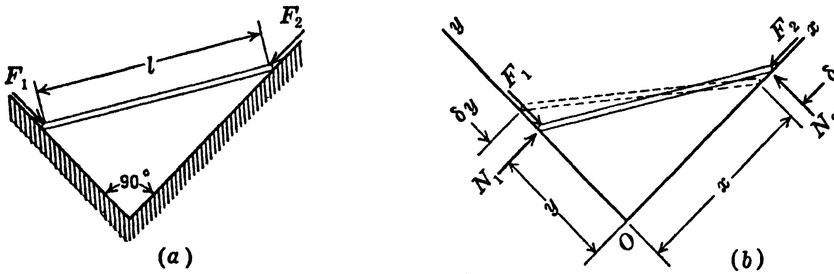


FIG. 3–5

Solution. A displacement of the system compatible with the constraints is shown in (b). From the geometry of the system, we have $x^2 + y^2 = l^2$, where l is a constant; we can find the relationship between δx and δy by differentiating:

$$2x\,\delta x + 2y\,\delta y = 0$$

$$\frac{\delta x}{\delta y} = -\frac{y}{x}$$

From the principle of virtual displacements, we have:

$$F_1\,\delta y - F_2\,\delta x = 0$$

$$F_1 = \frac{\delta x}{\delta y} F_2 = \frac{y}{x} F_2$$

where we omit the negative sign since we have already taken it into account by calling the work done by F_2 negative.

We have thus gone directly to the relationship between F_1 and

F_2 without considering or computing the reactions N_1 and N_2. If we had solved the problem by our previous equations of equilibrium, we would have included N_1 and N_2 in the equations and would then have eliminated them algebraically. It should be remembered, however, that in many problems it will be just as important to solve for the reactions as for the active forces.

Example 2. Find the equilibrium relationship between the force F_1, derived from the gas pressure in the engine cylinder, and the torque M about the crankshaft bearing O (Fig. 3–6).

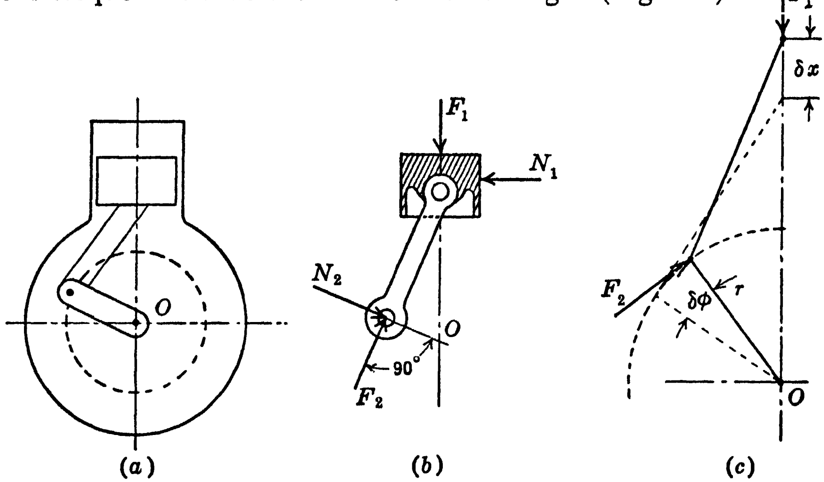


Fig. 3–6

Solution. The active forces in the system are F_1, and F_2 the force normal to the crank OA. A virtual displacement of the system is shown in (c), which involves a rotation of the crank about O, and a translation of the piston.

By the principle of virtual displacements we have:

$$F_2(r)\,\delta\phi - F_1\,\delta x = 0$$

$$F_2(r) = M,$$

so that

$$M = F_1 \frac{\delta x}{\delta \phi}$$

If we take the displacements as infinitesimal, we have:

$$M = F_1 \frac{dx}{d\phi}$$

We can find the relationship between x and ϕ from geometrical considerations in the form:

$$x = f(\phi)$$

so that the derivative could be found, and the relationship between M and F_1 would thus be known.

EXAMPLE 3. A compressive force P is applied by means of a screw-press as shown in Fig. 3–7. The screw is turned by a force F at a distance a from the axis. The pitch of the screw thread is b, where by pitch we mean the distance which the screw thread would advance, for one complete revolution of the screw. Find the relationship between F and P.

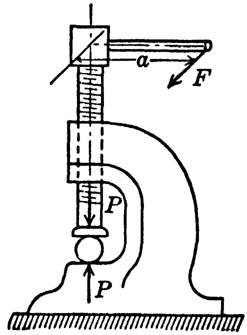


FIG. 3–7

Solution. We take as a virtual displacement of the system a small rotation $\delta\theta$ of the screw. The screw would then advance a distance $\frac{b}{2\pi}(\delta\theta)$. From the principle of virtual displacements we have directly:

$$F(a)\,\delta\theta = P\frac{b}{2\pi}\,\delta\theta$$

$$P = \frac{2\pi a}{b}F$$

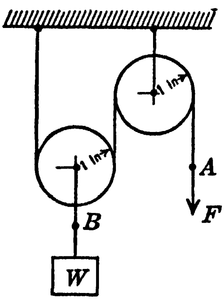


FIG. 3–8

EXAMPLE 4. A weight W is to be lifted by means of a force F applied to the system of two equal pulleys shown in Fig. 3–8. Find the relationship between F and W for equilibrium conditions.

Solution. As the virtual displacement of the system let us take a small vertical motion of the point A downward, δy. Then the point B will be raised by the amount $\frac{\delta y}{2}$, and the principle of virtual displacements gives:

$$F_1(\delta y) - W\left(\frac{\delta y}{2}\right) = 0$$

$$F = \frac{W}{2}$$

The Generality of the Principle of Virtual Displacements. From the above examples it will be seen that the advantage of the principle of virtual displacement lies in the over-all view which it gives us of the relationships of the external forces of a system. This principle will enable us in many cases to arrive at definite conclusions concerning the external forces acting on certain structures or machines, without a complete knowledge of the details of the arrangement. Suppose, for example, that we have a box with two levers protruding, as is indicated in Fig. 3–9.

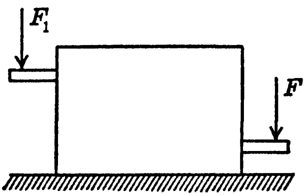


FIG. 3–9

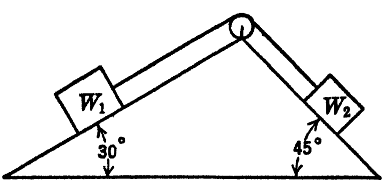
We apply two vertical forces to the levers as shown in Fig. 3–9, and we wish to know the relationship between the forces at equilibrium. If we find experimentally that a vertically downward movement of the left-hand lever of δy_1 produces a vertically upward movement of the right-hand lever of the amount δy_2, we can immediately say

$$F_1\,\delta y_1 - F_2\,\delta y_2 = 0; \quad \frac{F_1}{F_2} = \frac{\delta y_2}{\delta y_1}$$

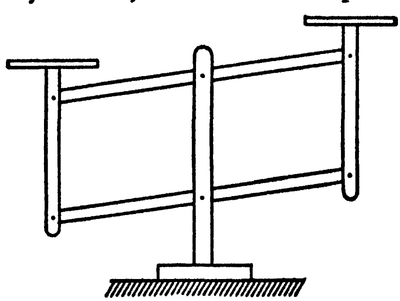
Thus we can find the relationship between the external forces while remaining in complete ignorance as to the details of the arrangement within the box.

PROBLEMS

77. Two weights rest on frictionless inclined planes as shown in the figure. They are connected by a perfectly flexible, inextensible cord pass-



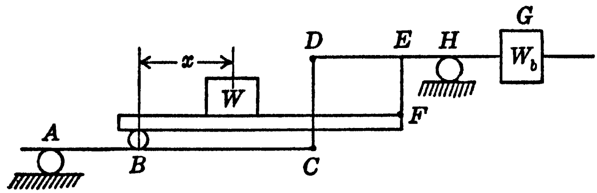
PROB. 77


PROB. 78

ing over a frictionless pulley. If W_2 weighs 150 lb, find, by the principle of virtual displacements, the magnitude of W_1 for equilibrium of

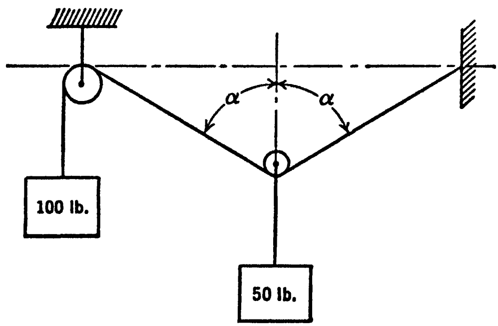
the system. As a comparison of the two methods, check this problem by writing the usual equations of equilibrium.

78. Show that a balance constructed as shown in the figure (Roberval's Balance) will give correct weight independent of the position of the weights on the scale pans.



PROB. 79

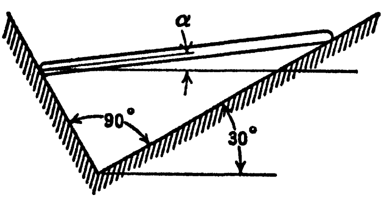
79. A platform scale is to be designed so that a weight W can be held in equilibrium by a weight W_b. How should the scales be constructed



PROB. 80

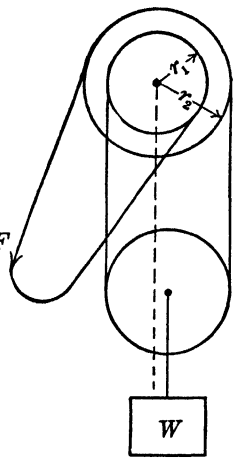
so that the readings will be independent of the position x of the weight on the platform?

80. Two weights are suspended on inextensible cords as shown in the diagram. What is the angle α for equilibrium?



PROB. 81

81. A uniform bar of weight W rests on two frictionless inclined planes as shown in the diagram. Show that the position of equilibrium of the bar is such that a virtual displacement would cause the center of gravity of the bar to move horizontally, and find the angle α for the equilibrium position.



PROB. 82

82. Find the force F required to keep the weight W in equilibrium, by means of the differential pulley arrangement shown in the diagram.

The Potential Energy of Systems in Equilibrium. We define the *potential energy* of a system * as follows: The change in the potential energy of a system which is moved from one configuration A to another configuration B is the negative of the work which is done by the forces of the system as the system is moved from A to B.

We can define a potential energy V only for those cases in which the forces acting on the system are functions only of the position of the system and are independent of the path followed by the system on going from one configuration to another. Consider the two situations shown in Fig. 3–10.

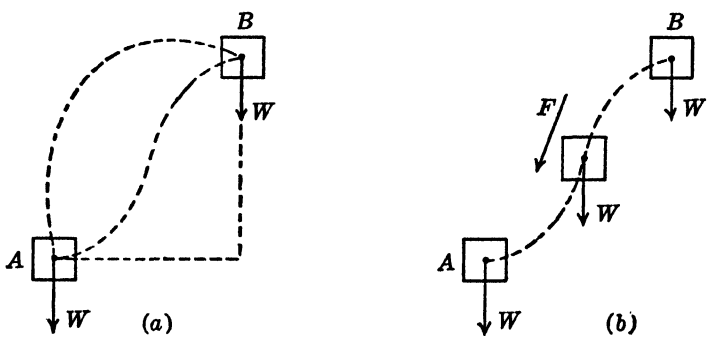


FIG. 3–10

In Fig. 3–10a we have shown a system acted upon only by gravity forces. The total work done by the forces during the motion from A to B will be independent of the path that is followed and will depend only upon the locations of the points A and B. In Fig. 3–10b, we have added air resistance forces to the system. In this case the work done will depend upon the path chosen, since more work would be done for a longer path. The air

* Here we consider only systems of masses.

resistance forces would also increase as the velocity of the body increases, so that the work done would thus also depend upon velocity. It will be seen that it is possible to define a potential energy as a unique determinate quantity only for cases such as (a). Such systems are said to be *conservative systems*. Systems of the type shown in (b), in which there are such effects as frictional dissipations which make it impossible to define a potential energy, are called *non-conservative systems*. We shall restrict ourselves in the following discussion to conservative systems.

With the above definition of a potential energy, the condition that the virtual work should disappear for an equilibrium position can thus also be stated as

$$\delta V = 0$$

i.e., for equilibrium of a force system the potential energy has a stationary value. Under various conditions, this stationary value might be a minimum, a maximum, or a point of inflection. We will now investigate the physical significance of these different possibilities.

Stable, Unstable, and Neutral Equilibrium. For simplicity, let us consider a system which is at rest at an equilibrium position, and which can move from that position of equilibrium in either direction by only one path.

Such a system is shown in Fig. 3–11, in which four different equilibrium configurations of a smooth cylinder are represented.

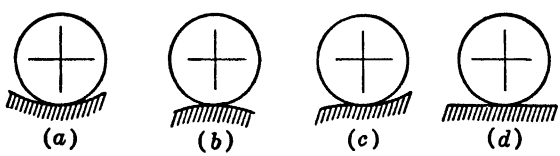


FIG. 3–11

In (a) the cylinder rests on a concave surface; in (b) on a convex surface; in (c) on a point of inflection; and in (d) on a plane surface. Applying the principle of virtual displacements, it will be seen that all four of the positions are equilibrium positions, since the only active force is a vertical gravity force, and since the center of gravity of the cylinder moves horizontally for an infinitesimal rolling displacement of the cylinder. We can also relate these situations to the fact that the potential energy of the system

must have a stationary value at the position of equilibrium. In (a), the potential energy of the system is seen to be a minimum, since the cylinder is in its lowest position, and work would have to be done on the cylinder to move it in either direction. In (b) the potential energy of the system is a maximum, since the cylinder is at the highest position, and work would be done by the system as it moved in either direction. The point of inflection of (c) corresponds to a stationary value, since the potential energy of the cylinder is neither a maximum nor a minimum.

It will also be recognized that the positions shown in Fig. 3–11 are physically somewhat different. In (a), we may say that we have *stable* equilibrium, since any movement away from the position of equilibrium sets up forces which will return the system to the equilibrium position, (b) and (c) are examples of *unstable* equilibrium, since any movement away from the position of equilibrium sets up forces which will move the system still further from the equilibrium position, and (d) illustrates *neutral* or *indifferent* equilibrium, in which motion of the system from the position of equilibrium does not affect the equilibrium of the system.

We can thus conclude that the maximum potential energy condition corresponds with the unstable condition, while the minimum potential energy condition corresponds with the stable condition.

The above considerations can be used to test an equilibrium position as to whether it is stable or unstable. It is only necessary to note whether the potential energy of the system increases or decreases as the system is moved from the position of equilibrium.

The conditions for equilibrium and the tests for stable and unstable equilibrium can be expressed analytically as follows. Suppose that for simplicity we consider a system whose displacement from a position of equilibrium can be described by one coordinate, x. Let P be this position of equilibrium, and V_P the potential energy of the system at the position P. Now, as x changes and we move away from the position of equilibrium, the potential energy of a conservative system will be some function of x:

$$V = f(x)$$

If we now write the Maclaurin series expansion of this function about the point P, we have:

$$V = V_P + x\left(\frac{dV}{dx}\right)_P + \frac{1}{2}x^2\left(\frac{d^2V}{dx^2}\right)_P + \cdots$$

$$V - V_P = x\left(\frac{dV}{dx}\right)_P + \frac{1}{2}x^2\left(\frac{d^2V}{dx^2}\right)_P + \cdots$$

By the theorem of virtual displacements, we know that the work done by the forces upon an arbitrary infinitesimal displacement from the point P is zero, if the point P is a position of equilibrium. Thus, if we choose an infinitesimal displacement dx, then the corresponding dV would be zero for equilibrium. The analytical condition for equilibrium of the system at the point P thus becomes:

$$\left(\frac{dV}{dx}\right)_P = 0$$

The change in the potential energy of the system, in the region near the equilibrium position, is then given by:

$$V - V_P = \frac{1}{2}x^2\left(\frac{d^2V}{dx^2}\right) + \cdots$$

since the term $\left(\frac{dV}{dx}\right)_P$ is zero.

Thus by examining the sign of the second derivative, we determine whether the potential energy is increasing or decreasing as we move away from the equilibrium position, and hence whether the system is stable or unstable.

From the above considerations, we see that while the positions of equilibrium can be determined by the principle of virtual displacements by considering only the first-order terms of the small displacements, it is necessary to investigate the second-order terms in order to decide as to the stability or instability of the equilibrium position.

Sign of $\left(\frac{d^2V}{dx^2}\right)_{x=0}$	Potential Energy V	Equilibrium Condition
+	minimum	stable
−	maximum	unstable

Example 1. A weight W_1 is supported by a rigid, weightless bar, AB, and a cable loaded by a weight W_2, as shown in

Fig. 3–12. Find the relationship between W_1 and W_2 for equilibrium of the system with the bar AB in a horizontal position, and determine whether this position of equilibrium is stable or unstable.

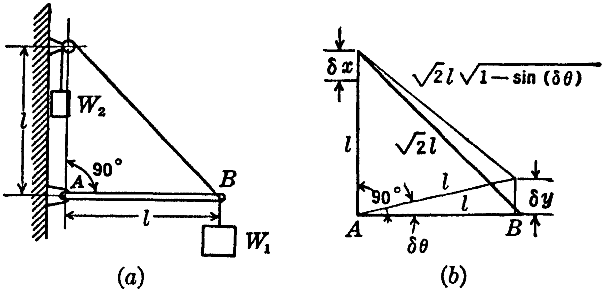


Fig. 3–12

Solution. We shall work this problem in two ways, first by the principle of virtual displacements, and second by means of potential energy considerations.

First Method

We take as a virtual displacement of the system a small lowering of the weight W_2, which we shall call δx. We can then find from the geometry of the system the distance through which W_1 rises, δy.

$$\delta x + \sqrt{2}\, l\sqrt{1 - \sin(\delta\theta)} = \sqrt{2}\, l$$

$$\sqrt{1 - \sin(\delta\theta)} = \frac{\sqrt{2}\, l - \delta x}{\sqrt{2}\, l}$$

$$\sin(\delta\theta) = \frac{2\sqrt{2}\, l(\delta x) - (\delta x)^2}{2l^2}$$

also

$$\sin(\delta\theta) = \frac{(\delta y)}{l}; \quad \delta y = l\sin(\delta\theta)$$

so

$$\delta y = \sqrt{2}(\delta x) - \frac{(\delta x)^2}{2l}$$

To find the equilibrium conditions, we note that the virtual work is zero, and that we can drop the second-order terms $(\delta x)^2$ in calculating this virtual work.

$$\delta W = W_2(\delta x) - W_1(\delta y) = 0$$

$$W_2(\delta x) - W_1[\sqrt{2}(\delta x)] = 0$$

$$W_2 = \sqrt{2}\, W_1$$

This answer can be very simply checked by equating the moments about the point A to zero, for equilibrium.

In order to investigate the stability of this equilibrium position, we must see whether it requires work to be done upon the system to move it from the position of equilibrium, or whether the system itself can do work as it moves from the position of equilibrium.

The work done by the system during the displacement δx from the position of equilibrium is:

$$\begin{aligned} & W_2(\delta x) - W_1(\delta y) \\ &= W_2(\delta x) - W_1\left[\sqrt{2}(\delta x) - \frac{(\delta x)^2}{2l}\right] \end{aligned}$$

Putting in the equilibrium condition $W_2 = \sqrt{2}\, W_1$ we have:

$$\begin{aligned} & \sqrt{2}\, W_1(\delta x) - \sqrt{2}\, W_1(\delta x) + W_1 \frac{(\delta x)^2}{2} \\ &= + W_1 \frac{(\delta x)^2}{2l} \end{aligned}$$

Note that we have left only a second-order term $(\delta x)^2$ so that second-order terms must be retained to investigate such stability problems.

Since this work term is positive, we see that the system itself does work as it moves from the position of equilibrium, i.e., the potential energy would be decreasing, hence the position of equilibrium is unstable.

Second Method

We write down the expression for the potential energy of the system in a region close to the equilibrium position:

$$V = V_P - W_2 x + W_1\left(\sqrt{2}\, x - \frac{x^2}{2l}\right)$$

where x is the downward displacement of W_2 from the equilibrium position, and V_P is the potential energy of the system at the equilibrium position, measured from any arbitrary level.

The equilibrium condition is:

$$\begin{aligned} \left(\frac{dV}{dx}\right)_{x=0} &= 0 = -W_2 + W_1\sqrt{2} - \frac{W_1 x}{l} \\ W_2 &= \sqrt{2}\, W_1 \end{aligned}$$

The test for stability is:

$$\left(\frac{d^2V}{dx^2}\right)_{x=0} = -\frac{W_1}{l}, \text{ negative, hence } \textit{unstable}$$

EXAMPLE 2. A uniform plank of thickness h is balanced on top of a circular cylinder of radius R (Fig. 3–13). What is the relationship between h and R for stability of the equilibrium position, assuming that the plank rolls on the cylinder without sliding?

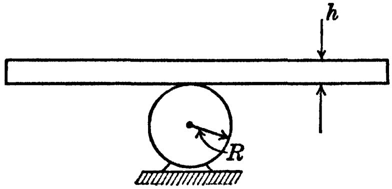


FIG. 3–13

Solution. Consider a virtual displacement $\delta\theta$ of the system consisting of the plank rolling on the cylinder (Fig. 3–14).

Originally, the points A and B coincide; after the virtual displacement the points A and B assume the positions shown in the

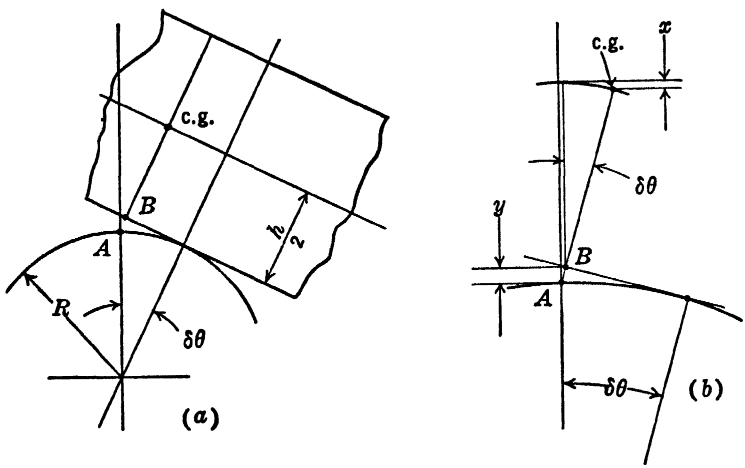


FIG. 3–14

diagrams. We wish to compute the total vertical movement of the center of gravity of the plank during the displacement $\delta\theta$. If this vertical movement is upward, the equilibrium position is stable. If this vertical movement is downward, the equilibrium is unstable. To solve for the limiting condition, we find the relationship between h and R for which there will be no vertical move-

ment of the c.g. In diagram (b) it will be seen that the total vertical motion can be considered as composed of two parts, one a downward component, due to the rotation of the plank, marked x in the diagram; and the other an upward component due to rolling of the plank, marked y in the diagram. The limiting condition for stable equilibrium will be determined by equating these two components.

From the figure, we have directly:

$$x = \frac{h}{2}[1 - \cos(\delta\theta)] = \frac{h}{2}\left[1 - 1 + \frac{1}{2}(\delta\theta)^2 - \frac{1}{24}(\delta\theta)^4 + \cdots\right]$$

By retaining second-order terms, but neglecting higher orders, we have:

$$x = \frac{h}{4}(\delta\theta)^2$$

To find y, note that in Fig. 3–15 the length y is less than AC, but greater than the vertical component of AD. We can show, however, that the difference between AC and $AD \cos(\delta\theta)$ involves only terms higher than second order, so that if we retain terms only through the second order we can say that $y = AC$.

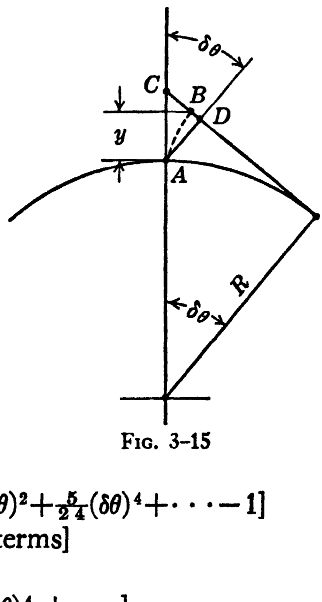


Fig. 3–15

$$\frac{R}{R + (AC)} = \cos(\delta\theta)$$

$$\begin{aligned}(AC) &= R[\sec(\delta\theta) - 1] = R[1 + \tfrac{1}{2}(\delta\theta)^2 + \tfrac{5}{24}(\delta\theta)^4 + \cdots - 1] \\ &= R[\tfrac{1}{2}(\delta\theta)^2 + \text{higher order terms}] \\ AD\cos(\delta\theta) &= R[1 - \cos(\delta\theta)]\cos(\delta\theta) \\ &= R[1 - 1 + \tfrac{1}{2}(\delta\theta)^2 - \tfrac{1}{24}(\delta\theta)^4 + \cdots] \\ &\qquad [1 - \tfrac{1}{2}(\delta\theta)^2 + \tfrac{1}{24}(\delta\theta)^4 + \cdots] \\ &= R[\tfrac{1}{2}(\delta\theta)^2 + \text{higher order terms}]\end{aligned}$$

Therefore, to the second-order approximation:

$$y = AC = \frac{R}{2}(\delta\theta)^2$$

For the limiting case of stable equilibrium,

$$x = y$$

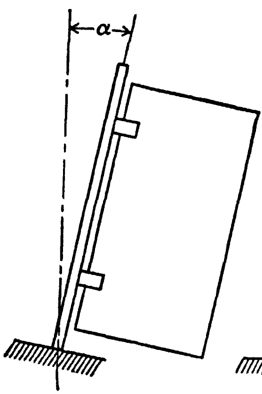
$$\frac{h}{4}(\delta\theta)^2 = \frac{R}{2}(\delta\theta)^2$$

$$R = \frac{h}{2}$$

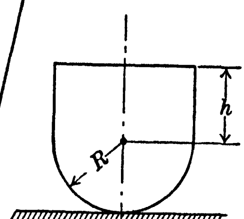
For example, a 2-in. thick plank would require a cylinder of 1-in. radius. Any smaller cylinder would represent a case of unstable equilibrium.

PROBLEMS

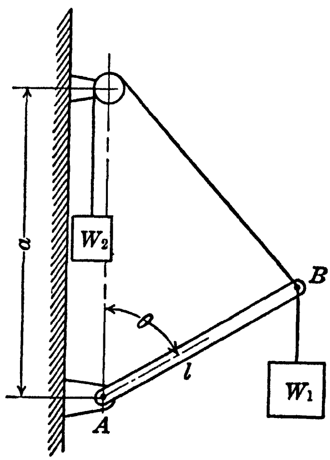
83. A door can swing through 360° around an axis which makes an angle α with the vertical, as shown in the figure. Show that the system has two positions of equilibrium, one of them stable, and the other unstable.



PROB. 83



PROB. 85



PROB. 86

84. Referring to Problem 81 above, determine whether the equilibrium position of the uniform bar is stable or unstable.

85. A homogeneous body is composed of a semi-cylinder and a rectangular parallelepiped, as shown in the figure. Find the maximum value of h consistent with stability of the system on the horizontal plane. The cylinder is assumed to roll on the plane without sliding.

86. A rigid weightless bar AB supports a weight W_1 and is supported by a cable which is loaded by a weight W_2 as shown in the figure. The distance a is greater than the distance l. Find all possible equilibrium positions of the system and investigate their stability.

CHAPTER IV

CENTER OF GRAVITY — DISTRIBUTED FORCE SYSTEMS — FLUID STATICS

For every heavy body there is a definite point called a center of gravity, in which point we may suppose the weight of the body collected.

—A Theorem of Archimedes.

To as good an approximation as is required in most engineering work, the system of gravity forces acting upon a body may be considered as a parallel force system. Since gravity forces are always present and must in most problems be taken into account, it will be desirable to simplify the treatment of such parallel force systems by means of a general analysis.

We shall first show that the system of parallel gravity forces acting upon a body has a resultant which passes through one particular point in the body, for any orientation of the body. This center of parallel gravity forces is called the *center of gravity* of the body, and the entire weight of the body can be considered as concentrated at that point, so far as statical considerations are concerned.

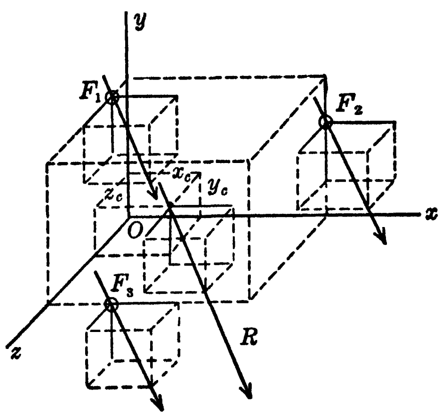


FIG. 4-1

Consider a rigid system which is represented by the three particles of Fig. 4-1.

The system is acted upon by parallel gravity forces F_1, F_2, and F_3. In this figure, the y-axis is not the vertical direction, but the system has been given some general orientation in space.

We now resolve each of the parallel forces into the three rectangular components:

$$F_x = k_1F$$
$$F_y = k_2F$$
$$F_z = k_3F$$

where k_1, k_2, k_3 are the direction cosines and are the same for all of the forces, since the forces are parallel.

Considering the system of forces parallel to the x-axis, we have by the principle of moments:

$$(R_x)y_c = \Sigma M_z = k_1F_1y_1 + k_1F_2y_2 + k_1F_3y_3$$
$$(R_y)z_c = \Sigma M_x = k_2F_1z_1 + k_2F_2z_2 + k_2F_3z_3$$

but

$$R_x = k_1F_1 + k_1F_2 + k_1F_3$$

so

$$y_c = \frac{k_1\Sigma(F)(y)}{k_1\Sigma F} = \frac{\Sigma(F)(y)}{\Sigma F}$$

and

$$z_c = \frac{\Sigma(F)(z)}{\Sigma F}$$

Similarly, taking force components parallel to the y-axis, we find:

$$x_c = \frac{\Sigma(F)(x)}{\Sigma F}$$

We have thus defined a point x_c, y_c, z_c, which is the point of application of the resultant of any parallel system of gravity forces. Since the particular direction cosines defining the orientation of the system cancel out of the analysis, it will be seen that the location of this center of parallel forces is independent of the orientation of the body. This point is called the *center of gravity* of the system.

Applying this principle to a homogeneous body of total weight W, we can reduce this single force W to a system of parallel forces each of magnitude dW so that $W = \int dW$. Then, if we locate each

element of weight dW by the coordinates x, y, z, we have for the location of the center of gravity of the body:

$$x_c = \frac{\int x\,dW}{\int dW}$$

$$y_c = \frac{\int y\,dW}{\int dW}$$

$$z_c = \frac{\int z\,dW}{\int dW}$$

The Center of Mass. Consider a system of particles having masses $m_1, m_2, \cdots$ as shown in Fig. 4–2, and let each particle be

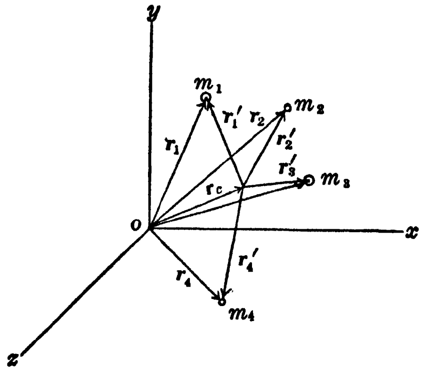


FIG. 4–2

located by its radius vector $\boldsymbol{r}$ from the origin of a rectangular coordinate system.

The *center of mass* of the system of particles is a point defined by the following expression:

$$\boldsymbol{r}_c = \frac{\Sigma m\boldsymbol{r}}{\Sigma m} \tag{1}$$

We can show that this point has the property that, if we multiply each mass by the vector from the center of mass to the particle, the sum of these moment products is zero, for:

$$\begin{aligned}\Sigma m\boldsymbol{r}' &= \Sigma m(\boldsymbol{r} - \boldsymbol{r}_c) = \Sigma m\boldsymbol{r} - \Sigma m\boldsymbol{r}_c \\ &= \boldsymbol{r}_c\Sigma m - \boldsymbol{r}_c\Sigma m = 0\end{aligned}$$

If we write r_c and r in terms of the rectangular components of the system, we have:

$$\boldsymbol{r}_c = x_c\boldsymbol{i} + y_c\boldsymbol{j} + z_c\boldsymbol{k}$$
$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$$

Substituting in equation (1) and equating components, we have:

$$x_c = \frac{\Sigma mx}{\Sigma m}$$

$$y_c = \frac{\Sigma my}{\Sigma m}$$

$$z_c = \frac{\Sigma mz}{\Sigma m}$$

Since the gravity force on the particle is proportional to the mass of the particle, it will be seen that the center of mass coincides with the center of gravity for all those cases in which the gravity forces can be assumed to form a parallel force system. Thus in engineering practice the terms *center of gravity* and *center of mass* are often used synonymously.

Centroids of Areas, Lengths, and Volumes. By generalizing the concept of a moment, the above ideas can be extended to several other important problems.

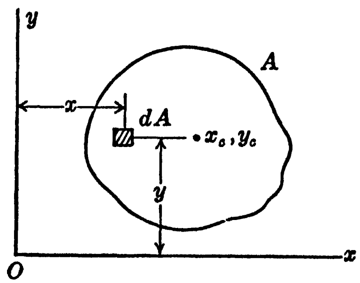


FIG. 4–3

Suppose that we have given an infinitesimal area, dA, located at a distance x from some axis. We then say that the *moment of the area* about the axis is the product $x\,dA$. To find the moment of any finite plane area about any line in its plane we have only to add up the moments of all of the elements, as shown in Fig. 4–3. The moment of dA about the y-axis is $x\,dA$, and the total moment of the whole area about the y-axis is:

$$M_y = \int x\,dA$$

We now define the distance x_c such that this distance multiplied by the total area will equal the moment of the area about the y-axis:

$$(x_c)(A) = \int x\,dA$$

In a similar way, the distance y_c is defined

$$(y_c)(A) = \int y\, dA$$

The point located by the coordinates x_c, y_c

$$x_c = \frac{\int x\, dA}{\int dA}$$

$$y_c = \frac{\int y\, dA}{\int dA}$$

is called the *centroid* of the area.

The quantity $\int x\, dA$ which we have called the moment of the area is often called the *first moment* of the area, and the expression $\int x^2\, dA$ is called the *second moment.* The name *moment of inertia of the area* is commonly used instead of the name second moment. A length r can be determined so that:

$$(A)(r^2) = \int x^2\, dA$$

This distance r is called the *radius of gyration of the area* about the y-axis.

These concepts of the moments of an area and of the centroidal distance and the radius of gyration will be found later to be useful in the analysis of a number of problems in the fields of dynamics and theory of elasticity.

The centroid of a length, a non-coplanar area, or a volume may be defined in the same way as we have defined the centroid of a plane area. Writing down the rectangular components of these centroids we have:

For the line of length l:

$$x_c = \frac{\int x\, dl}{\int dl}; \quad y_c = \frac{\int y\, dl}{\int dl}; \quad z_c = \frac{\int z\, dl}{\int dl}$$

For the surface area S:

$$x_c = \frac{\int x\, dS}{\int dS}; \quad y_c = \frac{\int y\, dS}{\int dS}; \quad z_c = \frac{\int z\, dS}{\int dS}$$

For the volume V:

$$x_c = \frac{\int x\,dV}{\int dV}; \quad y_c = \frac{\int y\,dV}{\int dV}; \quad z_c = \frac{\int z\,dV}{\int dV}$$

It will be seen that there is a close relationship between centroids and centers of gravity. The center of gravity of a homogeneous solid is located at the centroid of the volume of the solid. The terms *centroid* and *center of gravity* are used interchangeably in many books, although it is well to note that there are some cases, for example bodies with non-uniform specific weight distributions, in which the center of gravity might not coincide with the centroid of the volume.

The Computation of Centroidal Distances and of Moments of Inertia of Areas. In many cases the location of the centroid or center of gravity of a figure can be found by a consideration of the symmetry of the shape. If an area has an axis of symmetry, the centroid must be somewhere on this axis, and, if the area has two axes of symmetry, the centroid lies at the intersection of the two axes. Some figures which do not have an axis of symmetry may be said to have a center of figure, i.e., a point which is the midpoint of all elements of the figure which pass through the point. This point in many cases can be determined by inspection, as in the examples shown in Fig. 4–4.

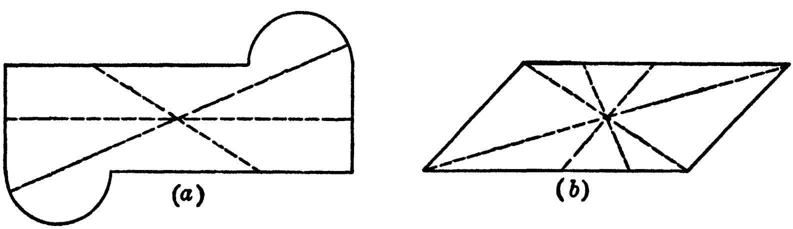


FIG. 4–4

The location of the centroids of a number of simple shapes can be found directly from the defining equations. For more complicated figures, the most useful scheme is to subdivide the shape into simple elements, for each of which the centroidal distances are known or may be easily determined. By summing up the moments of the individual elements, the centroid of the whole

figure is determined by the principle of moments. If it is not possible to subdivide the figure into simple figures for which the centroidal locations will be known, approximate solutions to any desired accuracy may be made by using sufficiently small elements.

The moments of inertia of many simple areas may be found directly by integration of the defining equations. For more complicated areas, the areas may be subdivided and the moments of inertias combined as discussed in the following paragraphs.

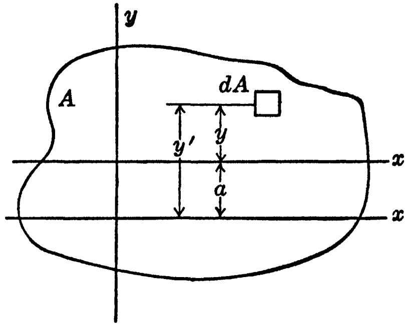


FIG. 4-5

The Transfer Theorem for Moments of Inertia. Suppose that the moment of inertia of an area is known about one line, and that the moment of inertia of the same area about some other line, parallel to the first line, is to be determined. In Fig. 4-5 the known moment of inertia about the line x is $I_x = \int y^2\,dA$ and the moment of inertia $I_{x'} = \int y'^2\,dA$ is to be determined. The constant distance between the two parallel lines is a. By definition:

$$I_{x'} = \int y'^2\,dA = \int (y+a)^2\,dA = \int y^2\,dA + 2a\int y\,dA + a^2\int dA$$

The first term $\int y^2\,dA$ is I_x, and the last term $a^2\int dA$ is a^2A. If we take the line x as passing through the centroid of the area, then the second term $2a\int y\,dA = 0$, and we have:

$$I_{x'} = I_{xc} + a^2A$$

where by the notation I_{xc} we indicate that the x-axis passes through the centroid of the area. This expression, which is called the transfer theorem for moments of inertia, states that the moment of inertia of any area is equal to the sum of the moment of inertia of the area about a parallel centroidal axis and a term equal to the area multiplied by the square of the distance to the centroidal axis. It should be particularly noticed that the parallel axis must be a centroidal axis, otherwise the second term in the

above analysis does not drop out, and the resulting expression has a more complex form. If the moment of inertia is known about some non-centroidal axis, and it is desired to find the moment of inertia about some other parallel non-centroidal axis, the parallel axis theorem must first be used to find the moment of inertia through the centroidal axis as an intermediate step in the problem.

Centroid locations and moments of inertia for a number of commonly encountered figures are given in Appendix II.

The Polar Moment of Inertia of an Area. It is also convenient to define a moment of inertia of an area about a line normal to the plane of the area. In Fig. 4–6 the xy axes are in the plane of

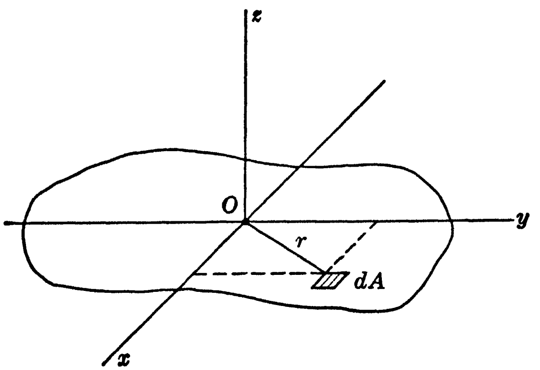


FIG. 4–6

the area; r is the distance from the z-axis to an element of area dA. *The polar moment of inertia of the area* is then defined as:

$$I_z = \int r^2 \, dA$$

We may also write $r^2 = x^2 + y^2$, so that:

$$I_z = \int x^2 \, dA + \int y^2 \, dA = I_y + I_x$$

Therefore the polar moment of inertia of an area about an axis through a point is equal to the sum of the moments of inertia of the area about two perpendicular axes in the area passing through the same point.

EXAMPLE 1. Find the location of the centroid of the volume of a hemisphere of radius r (Fig. 4–7).

Solution. We know that the centroid lies somewhere on the axis of symmetry of the hemisphere. Taking the axis of symmetry as the y-axis, the centroid can be located completely by finding y_c. Taking elementary slices of the hemisphere parallel to the xz plane, we have:

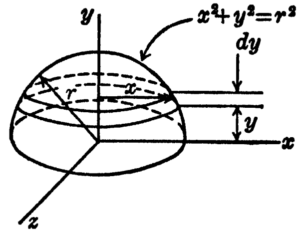


FIG. 4–7

$$y_c = \frac{\int y\,dV}{\int dV} = \frac{\int_0^r y(\pi x^2\,dy)}{\int_0^r \pi x^2\,dy} = \frac{\int_0^r y(r^2 - y^2)\,dy}{\int_0^r (r^2 - y^2)\,dy}$$

$$= \frac{\left(\frac{r^2y^2}{2} - \frac{y^4}{4}\right)\Big|_0^r}{\left(r^2y - \frac{y^3}{3}\right)\Big|_0^r} = \frac{\left(\frac{1}{2} - \frac{1}{4}\right)}{\left(1 - \frac{1}{3}\right)}\,r = \frac{3}{8}\,r$$

EXAMPLE 2. Find the moment of inertia of the area of a triangle about a line parallel to a base passing through the centroid of the area (Fig. 4–8).

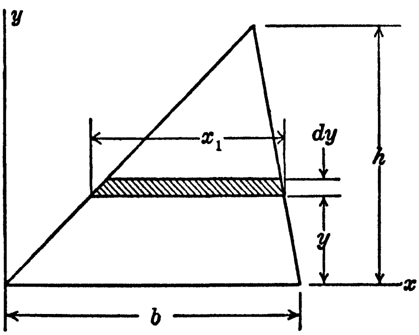


FIG. 4–8

Solution. We shall first find by direct integration the moment of inertia of the area of the triangle about the base.

$$I_x = \int y^2\,dA = \int_0^h y^2 x_1\,dy$$

$$= \frac{b}{h}\int_0^h (h - y)y^2\,dy$$

$$= \frac{b}{h}\left(\frac{hy^3}{3} - \frac{y^4}{4}\right)\Big|_0^h = \frac{1}{12}\,bh^3$$

Now, using the transfer theorem for parallel axes, we have:

$$I_x = I_c + \left(\frac{h}{3}\right)^2\left(\frac{1}{2}\,bh\right)$$

$$I_c = \tfrac{1}{12}\,bh^3 - \tfrac{1}{18}\,bh^3 = \tfrac{1}{36}\,bh^3$$

EXAMPLE 3. Find the centroid of the area shown in Fig. 4–9.

Solution. We first divide the figure into a number of simpler figures, each of which has a known centroidal distance. In the

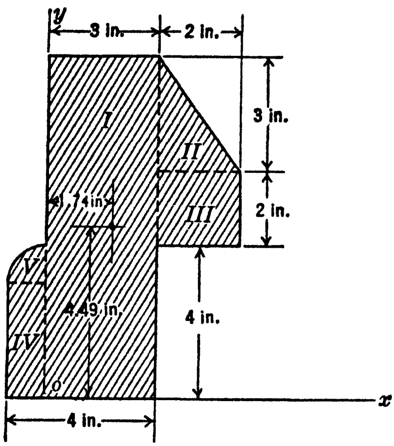


FIG. 4-9

present case we have enough pieces so that it will be desirable to systematize the calculations by placing them in a tabular form. Such tabular forms have a number of advantages. They are very easy to check because one knows at a glance just what the significance of each number on the sheet is, and the organization of complicated calculations in a tabulation form makes it possible to employ relatively unskilled computers for routine calculations.

Part	Area A (in.^{2})	Centroidal Dist.		$(A)(x)$ (in.^{3})	$(A)(y)$ (in.^{3})
		x (in.)	y (in.)		
I	27	1.5	4.5	40.5	121.5
II	3	3.67	7	11.0	21.0
III	4	4	5	16.0	20.0
IV	3	−0.5	1.5	−1.5	4.5
V	0.79	−0.42	3.42	−0.3	2.7
Σ	37.8			65.7	169.7

$$x_c = \frac{65.7}{37.8} = 1.74; \quad y_c = \frac{169.7}{37.8} = 4.49$$

EXAMPLE 4. Find the moment of inertia of the area shown in Example 3 about the x-axis.

Solution. The area is subdivided into the same elements as were used in the preceding problem. For each of these elements the moment of inertia about an axis through the centroid of the element parallel to the x-axis can be computed, by looking up the formula for the moment of inertia of the element in the appendix. The transfer theorem can then be used to find the moment of inertia of each element about the x-axis, and the sum of these

values will be the moment of inertia of the whole area. These calculations may be conveniently arranged in tabular form:

Part	I_c (in.^{4})	y_c (in.)	y_c^2 (in.^{2})	A (in.^{2})	Ay_c^2 (in.^{4})	$I_x = I_c + Ay_c^2$ (in.^{4})
I	182.2	4.5	20.3	27	548	730
II	1.5	7	49	3	147	149
III	1.33	5	25	4	100	101
IV	2.25	1.5	2.25	3	6.75	9
V	0.055	3.42	11.7	0.79	9.24	9
Σ						998

PROBLEMS

87. (a) Show that the area of the surface generated by rotating any plane curve about a non-intersecting axis in the plane of the curve is equal to the length of the curve times the distance traveled by the centroid of the curve.*

(b) Show that the volume of the solid generated by rotating any plane figure about a non-intersecting axis in the plane of the figure is equal to the area of the figure times the distance traveled by the centroid of the area.*

88. Find the center of gravity of a fine wire bent in the shape of a quarter-circle. Using this result, find the surface area of a sphere.

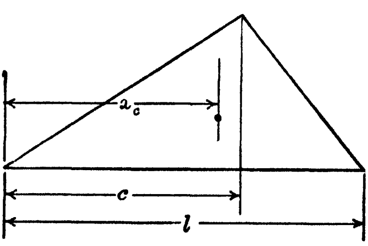
89. Find the centroid of the area under one-half cycle of a sine wave.

90. Find the centroid of a segment of a circle of radius r subtended by the angle α.

91. Show that the centroid of the volume of any cone or pyramid is located a distance of three-fourths of the altitude from the apex.

92. Show that for the triangle shown in the figure:

$$x_c = \frac{l + c}{3}$$



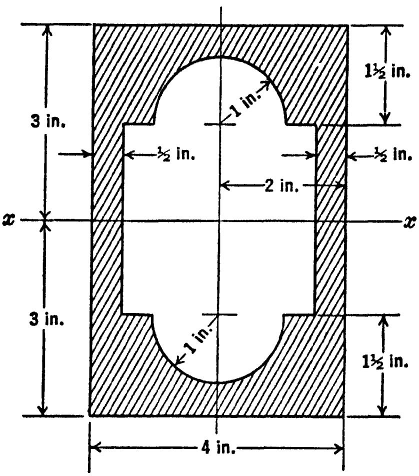
Prob. 92

93. A glass is partially filled with water. Show that the center of gravity of the glass and the water is lowest when this center of gravity is located in the surface of the water.

* These "theorems" are due to Pappus (A.D. 300), and were later independently noted by Guldinus (1635–1642).

94. Find the moment of inertia of the area of a circle with respect to a diameter of the circle.

95. Find the polar moment of inertia of the area of a rectangle with respect to an axis normal to the area passing through the centroid of the rectangle.



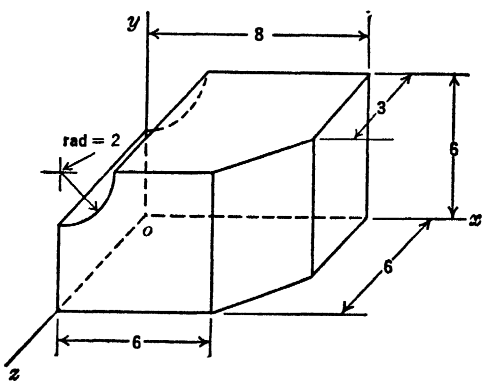
Prob. 97

96. Show that the polar moment of inertia of an area about any axis is equal to the sum of the polar moment of inertia of the area about the centroidal axis and the product of the area and the square of the distance between the axes.

97. Find the moment of inertia of the shaded area shown in the diagram about the axis *xx*.

98. Find the centroid of the volume shown in the figure.

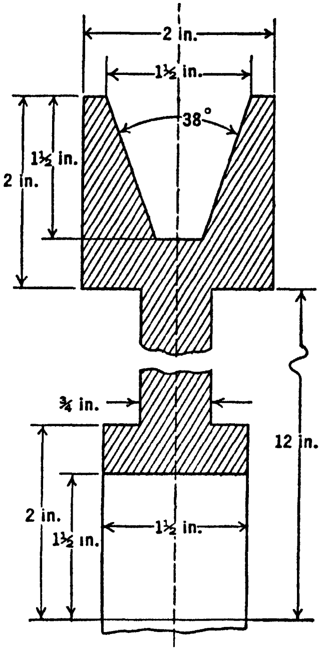
99. The cross section of a V-belt pulley has the dimensions shown in the figure. If the pulley is made of steel weighing 0.283 lb per cu in., find the weight of the pulley.



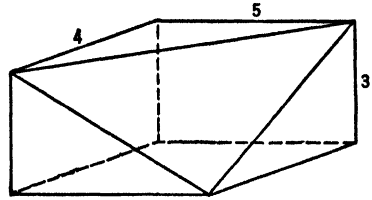
Prob. 98

100. Find the center of gravity of a homogeneous solid having the shape shown in the figure.

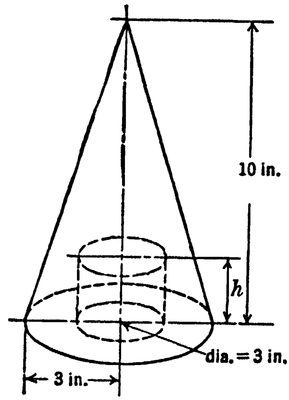
101. A 3-in. diameter hole is drilled into a cone as shown below in the figure. What should be the depth h of the hole if the centroid of the remaining volume is to be located a distance h from the base of the cone?



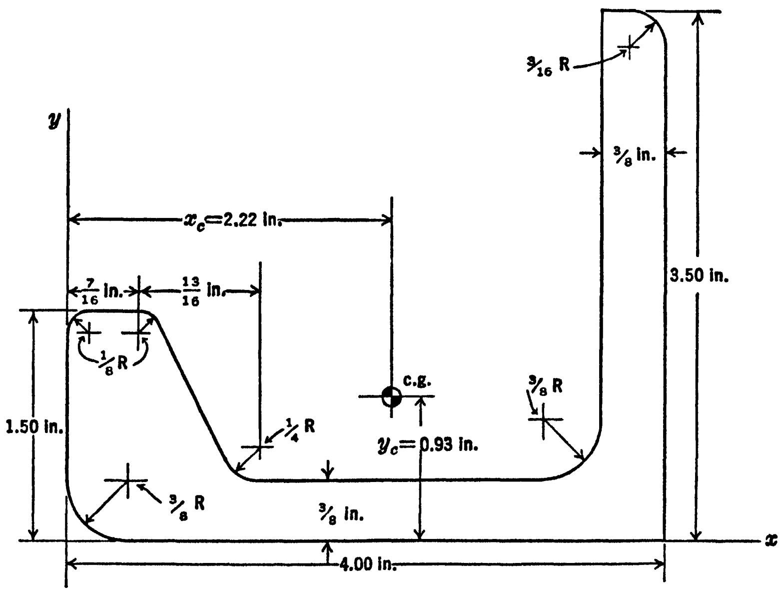
PROB. 99



PROB. 100



PROB. 101



PROB. 102

102. One standard size of the structural shape known as a bulb angle has the dimensions shown in the figure:

(a) The *Structural Aluminum Handbook* gives the following location for the center of gravity of the area: $x_c = 2.22$ in.; and $y_c = 0.93$ in. Check these values, neglecting the small areas represented by the fillets and rounded corners.

(b) The *Structural Aluminum Handbook* gives the moment of inertia of the area about the x_c-axis as 3.02 in.^{4} Check this value.

The Product of Inertia of an Area. Consider an area and an xy coordinate system in the plane of the area as shown in Fig. 4–10. The integral $I_{xy} = \int_A xy\, dA$ is called *the product of inertia of the area* with respect to the xy coordinate system. Integrals of this form often appear in the analysis of problems in dynamics and strength of materials and it will be necessary to evaluate such terms. For some simple figures it will be possible to compute the product of inertia directly by integration. For more complex figures, the figure may be broken down into simpler elements for which the products of inertia are known or may be easily determined, and the total product of inertia will then be the sum of the products of inertia of the parts. For this purpose it will be desirable to have a parallel-axis theorem for the transfer of the product of inertia from one coordinate system to another. Suppose that the products of inertia $I_{x_cy_c}$ of an area are known for rectangu-

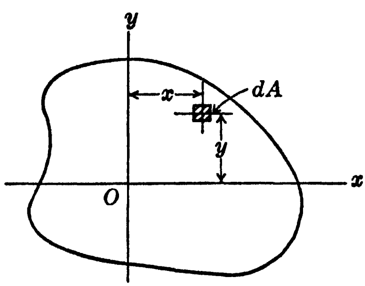


Fig. 4–10

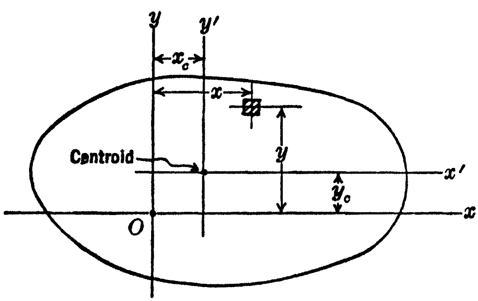


Fig. 4–11

lar axes located at the centroid of the area, as shown in Fig. 4–11. To find the product of inertia I_{xy} about any other parallel coordinate system, we have:

$$I_{xy} = \int_A (x_c + x')(y_c + y')\,dA$$
$$= \int_A x_c y_c\,dA + \int_A x' y_c\,dA + \int_A y' x_c\,dA + \int_A x'y'\,dA$$

The first term becomes $x_c y_c A$, the second and third terms drop out, since the $x'y'$ axes pass through the centroid of the area, and the last term is $I_{x_c y_c}$, so that:

$$I_{xy} = I_{x_c y_c} + x_c y_c A$$

If axes are to be transferred from one non-centroidal axis to a parallel non-centroidal axis, it should be noted that it will first be necessary to determine the product of inertia about a centroidal system as an intermediate step. The above expression for the transfer to parallel axes applies only when the $x'y'$ axes are at the centroid of the area.

Since the terms x and y appearing in the expression for the product of inertia may be either positive or negative, the product of inertia itself may be either positive or negative. That portion of the area which lies in the first quadrant will give a positive product of inertia, that portion which lies in the second quadrant will give a negative product of inertia, etc. The sign of the total product of inertia can usually be checked by inspection, since it will depend upon the relative portions of the area which lie in the various quadrants. If either the x or the y axis is an axis of symmetry for the figure, then $I_{xy} = 0$, since there will be a negative $xy\,dA$ term to cancel each positive $xy\,dA$ term. This fact can often be used, in combination with the parallel-axis theorem, to find the product of inertia without integration.

The Moment of Inertia of an Area about an Inclined Axis. When determining the moment of inertia of an area by integration it will usually be found that while it may be relatively easy to perform the integrations for certain positions of the axis, other positions may be difficult because of the shape of the boundary of the area. For example, the moment of inertia of a rectangular area may easily be found by integration for any axis which is

parallel to one of the sides. For an axis which is inclined at an angle to the sides, however, the integration and the substitution of the boundary conditions become more difficult. We should like, therefore, a method of determining the moment of inertia of an area about any inclined axis in an area, given the moment of inertia about some other axis.

In Fig. 4–12 suppose that the moments of inertia I_x and I_y about the xy axes are known, and that it is desired to find the moment

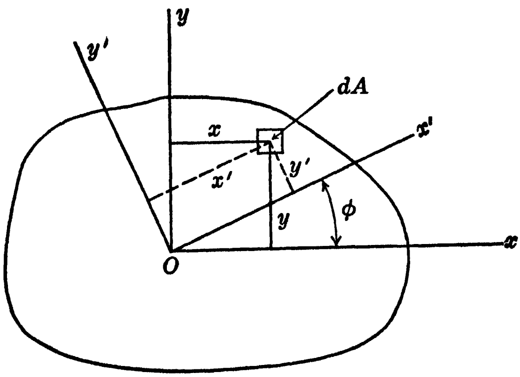


FIG. 4–12

of inertia $I_{x'}$ about the x'-axis which makes an angle ϕ with the x-axis. The coordinates of an element of area dA in the $x'y'$ system can be expressed in terms of the coordinates in the xy system.

$$x' = x \cos \phi + y \sin \phi$$
$$y' = y \cos \phi - x \sin \phi$$

The moment of inertia about the x'-axis is then given by:

$$\begin{aligned} I_{x'} &= \int_A y'^2 \, dA = \int_A (y \cos \phi - x \sin \phi)^2 \, dA \\ &= \int_A y^2 \cos^2 \phi \, dA - \int_A 2xy \sin \phi \cos \phi \, dA + \int_A x^2 \sin^2 \phi \, dA \\ &= I_x \cos^2 \phi + I_y \sin^2 \phi - I_{xy} \sin 2\phi \end{aligned}$$

Putting $\cos^2 \phi = \frac{1}{2}(1 + \cos 2\phi)$ and $\sin^2 \phi = \frac{1}{2}(1 - \cos 2\phi)$, this expression becomes:

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\phi - I_{xy} \sin 2\phi$$

Thus if the moments of inertia I_x and I_y, and the product of inertia I_{xy} are known, the moment of inertia about any inclined axis may be found.

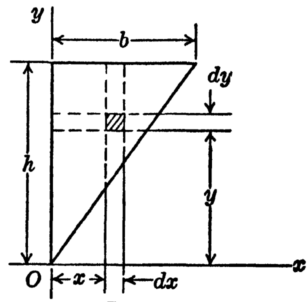


FIG. 4-13

EXAMPLE 1. Find the product of inertia of the area of a right triangle with respect to centroidal axes parallel to the base and altitude.

Solution. This problem may be solved most directly by first finding the product of inertia with respect to the xy system shown in Fig. 4–13 by integration, and then by using the transfer theorem to find the centroidal product of inertia.

$$I_{xy} = \int_0^h \int_0^{\frac{b}{h}y} xy\,dx\,dy = \int_0^h y\left(\frac{x^2}{2}\right)\Big|_0^{\frac{b}{h}y} dy = \frac{b^2}{2h^2}\int_0^h y^3\,dy = \frac{b^2}{2h^2}\frac{y^4}{4}\Big|_0^h$$
$$= \frac{1}{8}b^2h^2$$

From the transfer theorem, we have:

$$I_{x_cy_c} = I_{xy} - (x_c)(y_c)A = \tfrac{1}{8}b^2h^2 - (\tfrac{2}{3}h)(\tfrac{1}{3}b)(\tfrac{1}{2}bh)$$
$$= \tfrac{1}{8}b^2h^2 - \tfrac{1}{9}b^2h^2 = \tfrac{1}{72}b^2h^2$$

Note that the problem statement does not lead to a unique answer, since the sign of $I_{x_cy_c}$ depends upon the orientation (Fig 4–14).

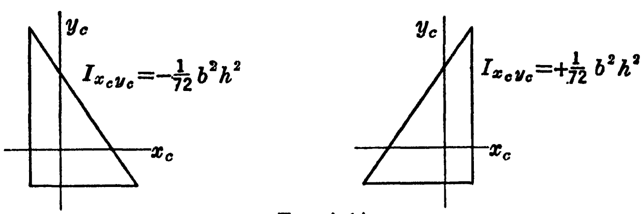


FIG. 4-14

EXAMPLE 2. Find the moment of inertia of the area of a rectangle about one of the long diagonals (Fig. 4–15).

Solution. We shall take the center of the rectangle as the origin of an xy coordinate system. In this system, we have:

$$I_x = \frac{ba^3}{12}$$
$$I_y = \frac{ab^3}{12}$$
$$I_{xy} = 0$$

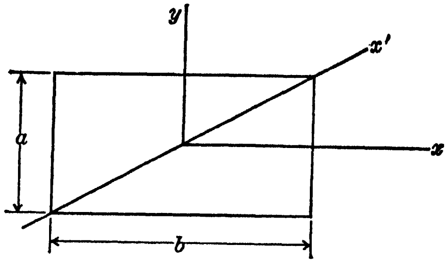


FIG. 4–15

The general expression for the moment of inertia about an inclined axis x' is:

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\phi - I_{xy}\sin 2\phi$$

In this case:

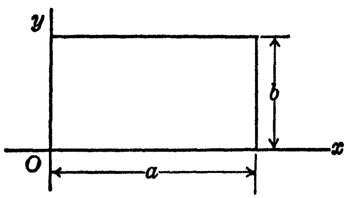
$$\cos 2\phi = 1 - 2\sin^2\phi = 1 - 2\left(\frac{a^2}{a^2 + b^2}\right) = \frac{b^2 - a^2}{a^2 + b^2}$$

So that:

$$I_{x'} = \frac{a^3b^3}{6(a^2 + b^2)}$$

PROBLEMS

103. Find by integration the product of inertia I_{xy} of the area of a rectangle which lies in the first quadrant as shown in the figure. Check this result by using the transfer theorem for products of inertia.



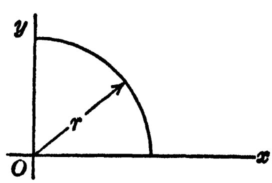
PROB. 103

104. Find the product of inertia I_{xy} of the area of a quarter circle, oriented as shown in the figure.

105. Find the product of inertia I_{xy} of the figure shown in Example 3, p. 87.

106. Show that the sum of the moments of inertia about two perpendicular axes in the plane of an area is a constant independent of the angle of rotation of the axes in the area, e.g., in Fig. 4–12, $I_{x'} + I_{y'} =$ constant. Show that this result would be expected from the definition of the polar moment of inertia.

107. Show that the product of inertia of an area with respect to two inclined axes (as in Fig. 4–12) is given by:



PROB. 104

$$I_{x'y'} = \frac{I_x - I_y}{2} \sin 2\phi + I_{xy} \cos 2\phi$$

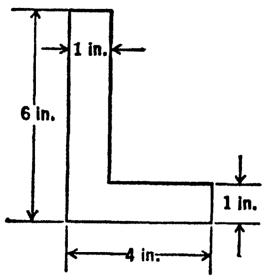
108. Show that the angle defining the axis about which the moment of inertia is a maximum or a minimum is given by (see Fig. 4–12):

$$\tan 2\phi = \frac{2I_{xy}}{I_y - I_x}$$

Show that if the moment of inertia is a maximum about one axis, then it is a minimum about the perpendicular axis. The axis about which the moments of inertia are maximum and minimum are called the *principal axes* of the area.

109. Show that the product of inertia of an area with respect to the principal axes is equal to zero (refer to Problem 108 above).

√**110.** The *Structural Steel Handbook* states that the least radius of gyration of the cross-sectional area of a 6 × 4 × 1 in. thick angle-iron section about any line in the area is 0.86 in. How closely is this value checked by assuming that the shape of the section is as given in the diagram? The value of this least radius of gyration is required for an analysis of the strength of such a structural member.



Prob. 110

Distributed Forces. Most of the forces with which we have been dealing have been assumed to be concentrated at one point. Our only exception, so far, has been the case of uniformly distributed gravity forces. There are a number of other common situations in which the forces are distributed throughout a volume or over an area. We know, for example, that it is impossible in nature to achieve a load which is concentrated at just one point. The inevitable deformation of loaded surfaces in contact will insure that the load is in actuality distributed over some small area, rather than being concentrated at a point. This distribution of load over an area is in general a desirable thing, since for engineering materials there is an upper limit to the load per unit area which can be safely supported, before a rupture or fracture of the material would occur. If a given load is to be supported by a surface, it is sometimes necessary to take special care to see that the load is distributed over an appreciable area, to avoid excessive unit loadings.

An example of such a loading is the beam shown in Fig. 4–16. In Fig. 4–16a the fact that the beam is supported at the two ends and loaded by a concentrated load between the supports is

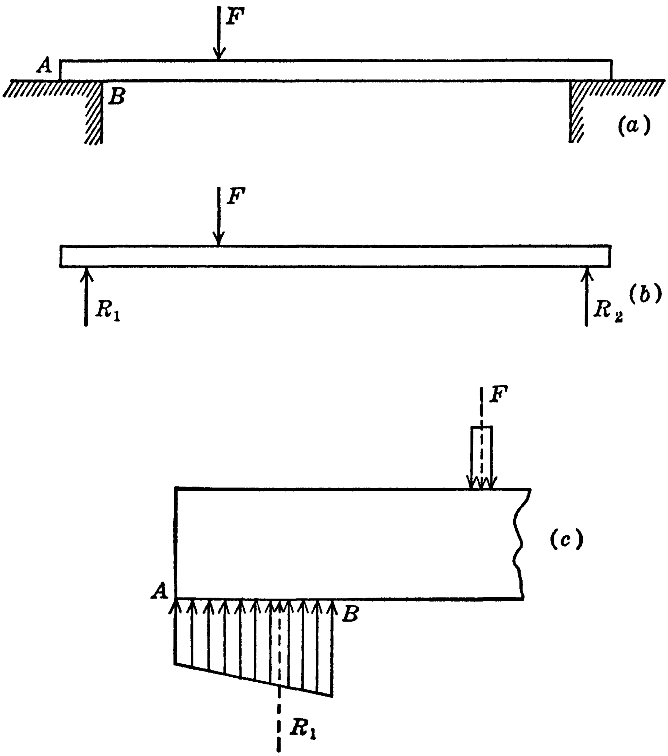


Fig. 4–16

indicated. In (b) the free-body diagram of the beam is shown, and the loads and reactions have been indicated, as in the preceding chapters, as concentrated loads. In (c) a more exact picture of the actual state of affairs is shown. Both the externally applied load F and the reactions R_1 and R_2 at the supports are in reality distributed systems of parallel forces. The force between the beam and the supporting surface is shown to be distributed over a length AB of the beam, instead of at a point. Because of the manner in which the beam has deformed, it is also likely that this force is non-uniformly distributed over the distance AB, and that it is greater at B than at A. We have accordingly shown, as

a first approximation, a linear variation of force along the length *AB*. This diagram, which indicates the way the force varies along the length of the beam is called the *force diagram*, and an ordinate of this diagram, a force per unit length, is called the *force intensity*. It will be seen that the area of the force diagram is equal to the resultant force of the parallel force system, and that this resultant passes through the centroid of the area of the force diagram. In this way we can reduce the distributed force system, for statical purposes, to a single force acting at a determinate point.

In the above example we have considered only those variations in the intensity of the force along the length of the beam. In some problems the variations of this force intensity across the width of the beam must also be considered. Such a three-dimensional force distribution is shown in Fig. 4–17.

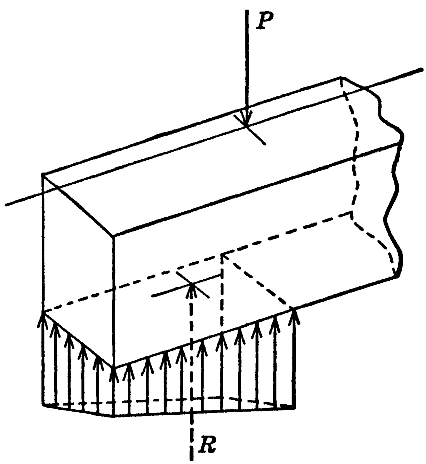


FIG. 4–17

In this case the load P is located off of the centerline of the beam, so that the reaction force intensities are greater along the back portion of the beam than they are along the front edge of the beam.

The reaction forces thus form, as in Fig. 4–17, a system of parallel forces in space. Instead of a force intensity, we now speak of a force per unit area, called a *pressure* or a *stress*. This nonuniformly distributed force system in space could thus be replaced, for statical purposes, by a single resultant force whose magnitude is equal to the volume of the pressure diagram, and which passes through the centroid of that volume.

EXAMPLE. One end of a beam is built into a wall as shown in Fig. 4–18a, while the other end is loaded by a concentrated force P. Assuming that the variation of the force intensity of the reaction forces on the beam is as shown in (b), find the maximum force intensities p_1 and p_2.

Solution. We can replace the distributed loads shown by resultant forces acting through the centroids of the force intensity diagrams, as shown in (c).

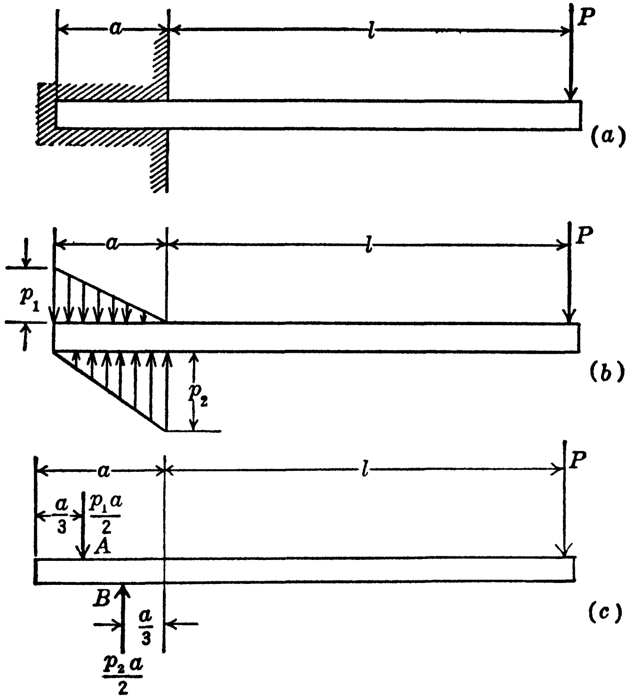


Fig. 4–18

Taking moments about the point A, we have for equilibrium:

$$\Sigma M_A = 0 = \left(\frac{a}{3}\right)\left(\frac{p_2 a}{2}\right) - \left(l + \frac{2}{3}a\right)P$$

$$p_2 \frac{a^2}{6} = \left(l + \frac{2}{3}a\right)P$$

$$p_1 = \frac{6P}{a^2}\left(l + \frac{2}{3}a\right)$$

also

$$\Sigma M_B = 0 = \left(\frac{a}{3}\right)\left(\frac{p_1 a}{2}\right) - \left(l + \frac{1}{3}a\right)P$$

$$p_1 \frac{a^2}{6} = \left(l + \frac{1}{3}a\right)P$$

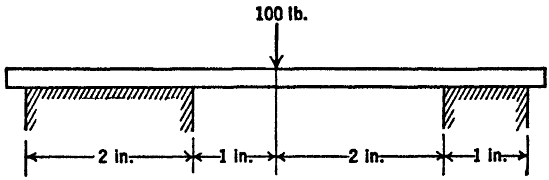
$$p_1 = \frac{6P}{a^2}\left(l + \frac{1}{3}a\right)$$

As a check equation, we can write the force summation equation:

$$\Sigma F_y = 0 = -\frac{p_1 a}{2} + \frac{p_2 a}{2} - P$$
$$= -\frac{3P}{a}\left(l + \frac{1}{3}a\right) + \frac{3P}{a}\left(l + \frac{2}{3}a\right) - P$$
$$= -\frac{3Pl}{a} - P + \frac{3Pl}{a} + 2P - P = 0$$

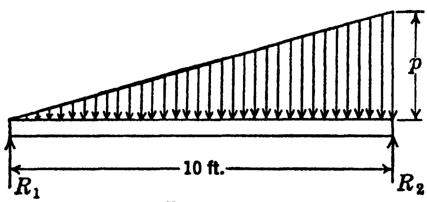
PROBLEMS

111. A beam supports a concentrated load of 100 lb and is supported as shown in the figure. Assuming that the reaction forces at the support are uniformly distributed over the width of the support, find the intensity of the force at the two supports.



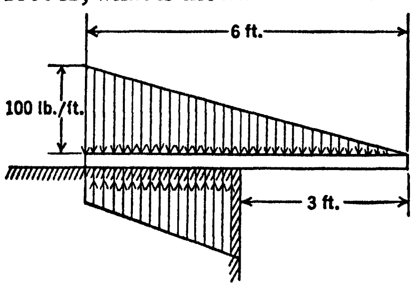
PROB. 111

112. The load on a beam varies linearly from zero at one end to a load p per unit length of the beam at the other end. If the reaction R at the heavily loaded end of the beam is not to exceed 1500 lb, what is the maximum total load that can be carried by the beam?



PROB. 112

113. A beam is loaded and supported as shown in the diagram. Assuming that the reaction force diagram has a trapezoidal form, find the reaction force intensities.



PROB. 113

Hydrostatics. By means of the principles of statics and the properties of distributed force systems as discussed above, a number of important relationships in the field of fluid statics can be established.

The Pressure in a Fluid. Consider a very small volume of fluid in the form of a triangular prism, as shown in Fig. 4–19.

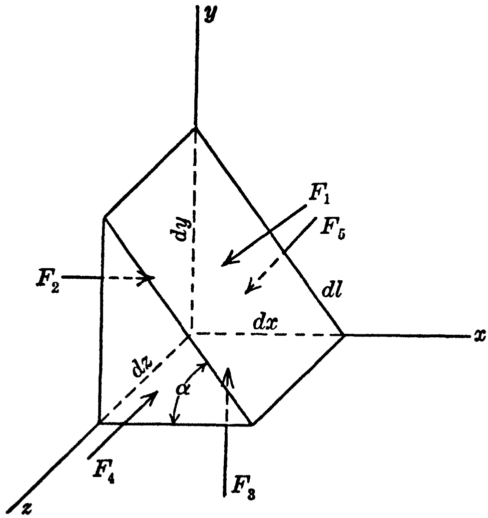


Fig. 4–19

From the definition of a fluid we know that under equilibrium conditions the forces F_1, F_2, etc., exerted on the faces of the prism by the fluid, are normal to the surfaces, i.e., there are no tangential forces in a fluid at rest. We define the *pressure* in the fluid as the surface force per unit area, and we shall denote such pressures by the small letter p. Hence in Fig. 4–19 the pressure p_1, associated with the force F_1 would be $\frac{F_1}{dz\,dl}$, assuming that the surfaces of the element are sufficiently small so that the force may be said to be uniformly distributed over the faces of the element.

Summing up the forces in the x and y directions, we have for the equilibrium condition:

$$\Sigma F_x = 0 = p_2\,dy\,dz - p_1\,dz\,dl\sin\alpha$$
$$\Sigma F_y = 0 = p_3\,dx\,dz - p_1\,dz\,dl\cos\alpha$$

In these equations we neglect any body forces proportional to the volume of the element, such as gravity forces, since as the element becomes infinitely small such body forces would involve small quantities of the third order, whereas the surface forces involve terms of the second order.

From the above equations, noting that $dl \sin \alpha = dy$ and $dl \cos \alpha = dx$, we have:

$$p_2\, dy\, dz - p_1\, dy\, dz = 0$$
$$p_3\, dx\, dz - p_1\, dx\, dz = 0$$
$$\therefore p_1 = p_2 = p_3$$

Since we have taken α as any angle, we can conclude that the pressure at a point in a fluid is the same on any surface passing through that point, independent of the direction of that surface. This is one of the fundamental differences between an ideal fluid and such bodies as an elastic solid, in which the pressure does depend upon the direction of the surface on which it acts.

The Pressure in a Fluid Under Gravity Forces. Consider an element of a fluid in the form of a vertical cylinder, located as shown in Fig. 4–20.

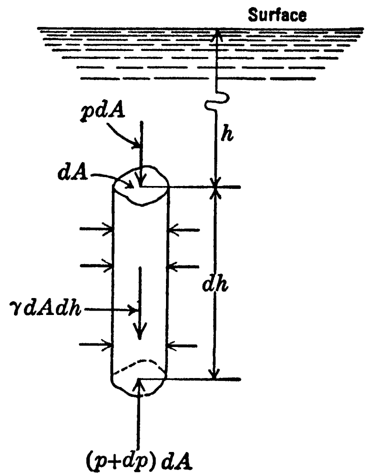


FIG. 4–20

If we draw a free-body diagram of this element in equilibrium, we will have acting the forces shown; a force $p\, dA$ acting vertically downward on the top of the element, an upward force $(p + dp)\, dA$ acting on the bottom of the element, and a gravity force equal to the weight of the fluid in the element. Using the notation:

ρ = the density of the fluid = the mass per unit volume
$\gamma = \rho g$ = the specific weight of the fluid = the weight per unit volume, where g is the acceleration of gravity

the weight of the fluid in the element is:

$$\gamma\, dA\, dh$$

For equilibrium, we have:

$$\Sigma F_y = 0 = (p + dp)\, dA - p\, dA - \gamma\, dA\, dh$$
$$dp - \gamma\, dh = 0$$
$$\frac{dp}{dh} = \gamma$$

From this relationship we can determine the variation of pressure with depth in an incompressible fluid, for which γ is a constant:

$$\int_{p_1}^{p_2} dp = \gamma \int_{h_1}^{h_2} dh$$

$$(p_2 - p_1) = \gamma(h_2 - h_1)$$

hence the difference in pressure at two points varies directly with the difference in the depth of the two points.

Hydrostatic Force on Plane Areas. Let us find the total force due to fluid pressure upon a submerged plane area A. We will assume that at the surface of the fluid the pressure is zero (neglecting atmospheric pressure) and we will measure the depth h vertically downward from this free surface.

From the preceding equation, putting $p_1 = 0$ when $h = 0$

$$p = \gamma h$$

Considering first the horizontal plane shown in Fig. 4–21a, we see that the pressure has everywhere on this plane the same

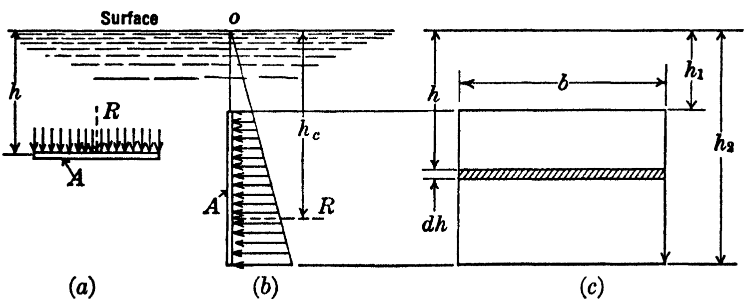


FIG. 4–21

value γh, so that the total downward force acting on the plane is γhA, acting at the centroid of the area. In the case of the vertical area, however (Fig. 4–21b), the pressure varies along the depth of the area, and the system of forces acting upon the area becomes a system of non-uniform parallel forces. The resultant of this non-uniform system of parallel forces passes through the centroid of the pressure diagram, rather than through the centroid of the area. The point on the submerged body at which the resultant of the pressure force acts is called the *center of pressure.*

As an example, suppose we have a rectangular area, as shown in Fig. 4–21c. The total force on the area and the location of the center of pressure can be found by integration; the force on the shaded element of area is:

$$\gamma hb(dh)$$

Summing up these forces we have for the total force R:

$$R = \gamma b \int_{h_1}^{h_2} h\,dh = \gamma b \left(\frac{h^2}{2}\right)\Bigg|_{h_1}^{h_2}$$

$$R = \frac{\gamma b}{2}(h_2^2 - h_1^2)$$

To find the center of pressure, take moments about the point O:

$$(h_c)(R) = \gamma b \int_{h_1}^{h_2} h^2\,dh = \gamma b \left(\frac{h^3}{3}\right)\Bigg|_{h_1}^{h_2}$$

$$h_c = \frac{2}{3}\frac{(h_2^3 - h_1^3)}{(h_2^2 - h_1^2)}$$

(for a rectangular area)

Hydrostatic Force on Any Curved Area. Buoyant Forces. The above considerations can be extended to include curved areas:

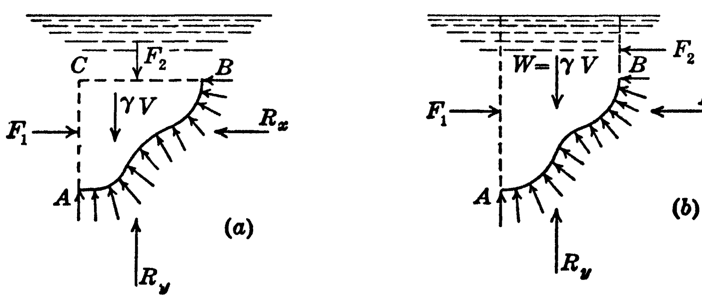


FIG. 4–22

Let AB be the edge view of a cylindrical surface whose length perpendicular to the paper is unity. We wish to find the horizontal and vertical components R_x and R_y of the system of forces acting on one side of this surface due to the fluid.

In Fig. 4–22a imagine a fluid element bounded by AB, a horizontal plane BC, and three vertical planes. For equilibrium of this element, we have:

$$\Sigma F_x = 0 = F_1 - R_x$$

$$R_x = F_1$$

Hence the magnitude and the location of the horizontal component will be the same as the force acting upon the vertical plane *AC* and can be computed as in the preceding paragraph.

To find the vertical component, we imagine that the element extends to the free surface, as in Fig. 4–22b. The force on the top of this element will then be zero, and the condition for equilibrium is:

$$\Sigma F_y = 0 = R_y - \gamma V$$
$$R_y = W$$

Hence the vertical component of force is equal in magnitude to the weight of the fluid contained in the vertical column above the area, and passes through the center of gravity of that fluid element. It will be seen that this same result would be obtained for an area element of any kind.

If we apply the above principles to the case of a closed area, such as the surface of a submerged body, we are led directly to Archimedes' Principle for a floating body.

Consider the forces acting upon a cylindrical element of a submerged body, as shown in Fig. 4–23. Considering the area at *A*,

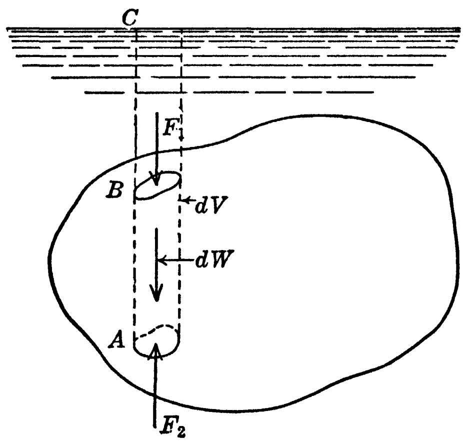


FIG. 4–23

there will be acting at *A* a force F_2 which is equal to the weight of the fluid in the column *AC*. There is also acting at *B* a downward force F_1, which is equal to the weight of the fluid in the column *BC*. There will thus be a net upward force F_2–F_1 acting on the element which will be equal to the weight of the fluid con-

tained in the volume dV, i.e., the volume of the column AB. This net upward force upon the element is called a *buoyant force.* If we now add up the buoyant forces on all of the elements of the body, we find that the total buoyant force acting upon the body is equal to the weight of the fluid displaced by the body.

The Equilibrium and Stability of Floating Bodies. When a body floats in equilibrium in a fluid, the buoyant force of the fluid is just equal to the gravity force on the body. A floating body thus submerges until an amount of fluid is displaced which weighs just as much as the body.

The question as to whether or not this position of equilibrium is stable, is of great importance in the design of ships. It will be seen at once that such floating equilibrium positions are stable so far as vertical displacements are concerned. If, for example, the body should be moved slightly downward from the equilibrium position into the fluid, more fluid will be displaced and the upward buoyant force will increase, thus tending to move the body back to the equilibrium position. Similarly, if the body is raised, the decrease in buoyant force will result in a net downward force which will tend to move the body back into the equilibrium position. So far as horizontal movements are concerned, the equilibrium will be seen to be neutral, since the forces of the system are not changed during such displacements. The question of stability of the body in the presence of angular motions, however, cannot be settled so quickly, and requires further consideration.

In Fig. 4–24a is shown the cross section of a ship floating in the water. The center of gravity of the ship is at the point G, and the center of gravity of the displaced fluid, shown shaded, is at

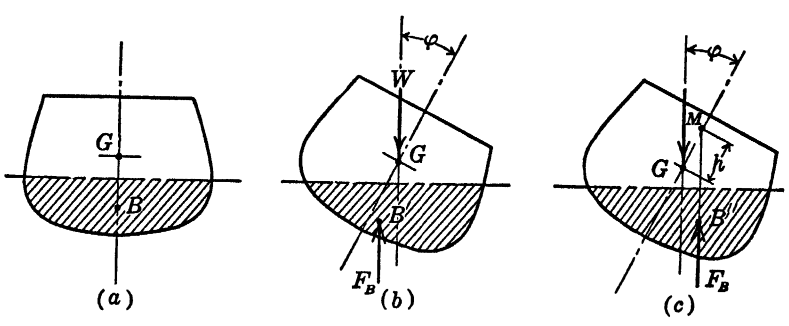


Fig. 4–24

the point B, in the equilibrium position. The point of application of the buoyant force F_B is at the point B. If now, we rotate the ship about a longitudinal axis through a small angle ϕ, the point B will assume some new position, such as B' or B'' in Fig. 4–24b and 4–24c. Depending upon the position of this point B, two different situations will exist. In Fig. 4–24b it will be seen that for the position of B' shown there, the forces F_B and W form a couple which has such a direction that it tends to increase the angle of roll. The equilibrium is therefore in this case unstable. In Fig. 4–24c, on the other hand, the position of B'' is such that a couple is formed which tends to return the ship to the equilibrium position, and hence the equilibrium position is, in this case, stable. Since the position of the points B' and B'' evidently depends upon the geometry of the system, it is clear that for each floating body an investigation of such stability conditions must be made.

Referring to Fig. 4–24c we observe that the line of action of the buoyant force F_B intersects the line through the points G and B at the point M. This point is called the *metacenter* of the body, and the distance h between the metacenter and the center of gravity is called the *metacentric height*. It will be seen that if the metacenter is above the center of gravity, the equilibrium is stable, whereas, if the metacenter is below the center of gravity, the equilibrium is unstable.

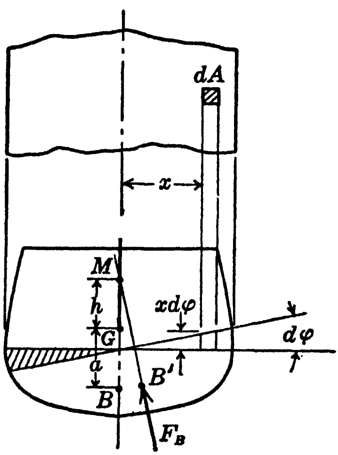


FIG. 4–25

The metacentric height can easily be determined experimentally for any particular ship. Suppose that for the ship shown in Fig. 4–24 we find experimentally that a weight w placed a distance b from the centerline of the ship causes the ship to roll through some small angle ϕ. From the equilibrium conditions we would have:

$$wb \cos \phi = Wh \sin \phi$$

$$\text{for } \phi \text{ small; } \cos \phi \approx 1, \ \sin \phi \approx \phi$$

$$\therefore h = \frac{wb}{W\phi}$$

The metacentric height can also be computed, as we shall now show. In Fig. 4–25 suppose that the surface of the water is rotated through an infinitesimal angle $d\phi$, which is equivalent to having the ship rotate through the angle $d\phi$. As a result of this rotation the center of buoyancy will shift from B to B' and this shift can be computed by taking moments of the volumes about the point B, calling the displaced volume V.

$$(V)(0) + \int_A (x\, d\phi)\, dA \cdot x = (V)(BB')$$

$$d\phi \int_A x^2\, dA = (V)(BB')$$

$$BB' = \frac{d\phi}{V} \int_A x^2\, dA$$

The integral $\int_A x^2\, dA$ is summed up over the whole water-line area of the ship. Denoting this moment of inertia of the water-line area of the ship by I, we have:

$$BB' = \frac{d\phi}{V} I$$

Also from Fig. 4–25, we have:

$$BB' = (h + a)\, d\phi$$

Thus: $(h + a)\, d\phi = \frac{I}{V}\, d\phi$, and since $V = \frac{W}{\gamma_F}$

$$h = \frac{\gamma_F}{W} - a$$

EXAMPLE. A rectangular parallelepiped having a square base floats in a fluid with the square base horizontal. The material of the block is homogeneous and has a specific weight γ. What is the maximum height of the block consistent with rolling stability?

Solution. We find the maximum height t, which is consistent with stability, by equating the metacentric height to zero.

$$h = \frac{\gamma_F}{W} - a = 0$$

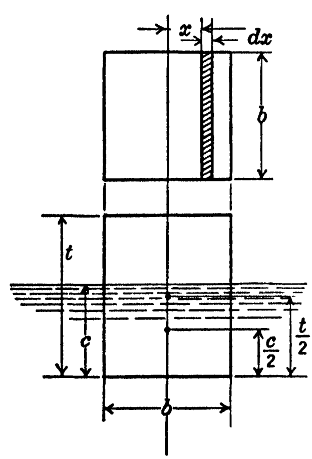


FIG. 4–26

a is the distance between the center of gravity of the body and the center of buoyancy, and hence is equal to:

$$a = \frac{t}{2} - \frac{c}{2} = \frac{1}{2}(t - c)$$

If γ_F is the specific weight of the fluid, we have:

$$b^2 t\gamma = b^2 c\gamma_F$$

$$c = \frac{\gamma}{\gamma_F} t$$

hence

$$a = \frac{1}{2}\left(t - \frac{\gamma}{\gamma_F} t\right) = \frac{t}{2}\left(1 - \frac{\gamma}{\gamma_F}\right)$$

To evaluate the moment of inertia I, we indicate the element of area of the water-line area as $b\,dx$, the integral becomes:

$$I = 2b\int_0^{\frac{b}{2}} x^2\,dx = 2b\left(\frac{x^3}{3}\right)\Bigg|_0^{\frac{b}{2}}$$

$$I = \frac{b^4}{12}$$

Thus we have:

$$\frac{t}{2}\left(1 - \frac{\gamma}{\gamma_F}\right) = \frac{\gamma_F b^4}{12b^2 t\gamma}$$

$$t = b\sqrt{\frac{1}{6\left(1 - \frac{\gamma}{\gamma_F}\right)\left(\frac{\gamma}{\gamma_F}\right)}}$$

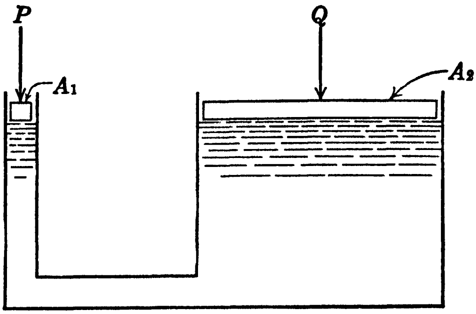
As a specific example, take a block of wood which weighs 40 lb per cu ft and float it in water having a weight of 62.4 lb per cu ft. Then the maximum height consistent with rolling stability is:

$$t = b\sqrt{\frac{1}{6\left(1 - \frac{40}{62.4}\right)\left(\frac{40}{62.4}\right)}} = 0.851b$$

e.g., if $b = 4$ in., $t_{max} = 3.4$ in.

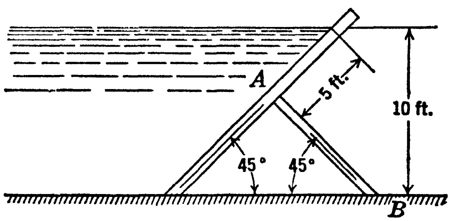
PROBLEMS

114. Find the equilibrium relationship between the loads P and Q on the hydraulic press shown in the diagram. The areas of the two pistons are A_1 and A_2, and the weights of the pistons may be neglected.



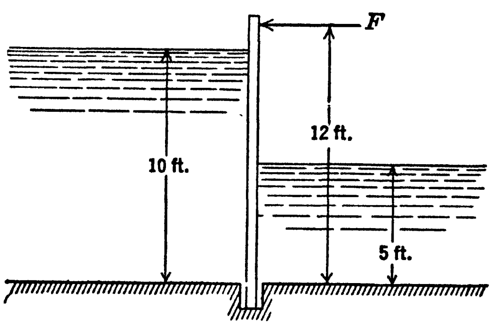
PROB. 114

115. The supporting members AB for the framed dam shown in the figure are placed 5 ft apart along the length of the dam. What is the force in AB?



PROB. 115

116. A sluice-gate is supported by a force F as shown in the figure. What is the magnitude of F for the water levels shown?



PROB. 116

117. A cube of homogeneous material floats in water. What should be the specific weight of the material if the system is to be stable for rolling motions?

CHAPTER V

FRICTION: THE APPLICATION OF STATICS TO MACHINE DESIGN

The physical cause of the resistance offered by friction to the motion of surfaces which slide on each other can be explained either by supposing an interlocking of the roughnesses of the surfaces, which cannot be separated unless they yield or break or are lifted over the tops of one another; or by supposing that the molecules of the surfaces of the two planes in contact are so close together that they develop a cohesion which must be overcome to produce the motion; experiment alone can enable us to determine the reality of these different causes.
—Coulomb, *Théorie des Machines Simples* (1779).

For the present chapter we shall consider the application of the principles of statics to the solution of a number of problems taken from the field of machine design.

An analysis of the forces involved in the various members of a machine is one of the first steps in the design of any machine, and we shall find that the principles of statics, as we have developed them above, play a very important part in this force analysis.

In many of the problems which we have solved, we have stated that the forces between surfaces in contact are normal to the surfaces in contact, i.e., that the reactions were frictionless. We now wish to investigate more exactly the true nature of these reactions, and to include in our analysis the effects of frictional forces which in all physical problems must be present in some degree.

These frictional forces will sometimes represent undesirable elements of a physical situation, in that there will be associated with them an energy dissipation. At other times we may wish to take advantage of the frictional forces, as in the case of brakes, clutches, and some types of propulsion systems.

Dry Friction. Consider a block resting on a plane, as shown in Fig. 5–1. We suppose that the surfaces in contact are clean and dry, and that no lubricating substances are present.

If we apply some very small force P to the block, our experience tells us that under ordinary conditions the block would not move. This means that there must have been set up at the contact surfaces some reaction force F, which just balances the applied force P. This force F is physically possible because the surfaces in contact are not absolutely smooth, and, because of small irregularities, a reaction force having a component parallel to the surfaces can be developed. Our experience further tells us that if we increase the load P we will finally reach a value which will cause the block to move, since it will no longer be possible for a frictional force of sufficient magnitude to be established. This maximum value of frictional force which it is possible to establish, under a given set of conditions, is called the *limiting friction* and a system which is acted upon by this limiting frictional force is said to be in a state of *impending* motion, since any further increase in the applied forces would cause motion.

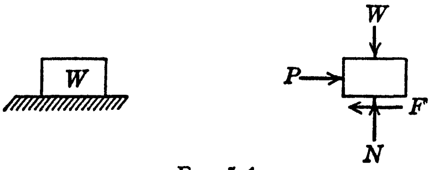


Fig. 5-1

The magnitude of this limiting frictional force has been experimentally determined for a large number of different kinds of contact surfaces, and for various conditions of contact area and normal load.

These experimental facts have been embodied in the laws of dry or Coulomb friction, first stated in a complete way by Coulomb.

(1) The total amount of friction is independent of the magnitude of the area in contact.
(2) The total amount of friction is proportional to the normal force.
(3) For low velocities of sliding the total amount of friction is independent of velocity, and is less than that for impending motion.

These laws are summed up analytically by the following defining equation:

$$F = \mu N$$

where F = total amount of frictional force parallel to the surfaces in contact, at impending motion, or during motion

N = total normal force between the surfaces in contact

μ = coefficient of friction, which is a constant for given materials

The actual mechanism of dry friction is a complicated one, so that the above laws should be considered as approximations which are sufficiently accurate for most engineering applications.

If the value of F in the above equation is the limiting value of friction at impending motion, then μ is called the *coefficient of static friction.* It is important to note that the use of this coefficient of static friction gives the maximum magnitude of the frictional force that it is possible to develop between two surfaces under a given normal load and does not necessarily give the actual frictional force present, which may be less than this maximum possible limit.

The following table will indicate some values of the coefficient of static friction which may be obtained under various conditions. It is only possible to indicate a range of values in such a table, since the surface condition plays an important part in the process. It should be emphasized that if any more exact determinations of frictional forces are required for a specific application, it will usually be necessary to determine experimentally the coefficient of static friction under conditions which are as nearly similar as possible to the actual application.

Coefficients of Static Friction

Wood on wood	0.25–0.50
Metal on wood	0.20–0.60
Metal on metal	0.15–0.30
Metal on leather	0.30–0.60
Wood on leather	0.25–0.50

In the event that there is sliding motion between the bodies, the *coefficient of kinetic friction* is used to find the friction force. This coefficient of kinetic friction depends so much, however, upon the velocity, the presence of material which may act as a lubricant, etc., that it is not possible to give even approximate values in the form of a table.

Since the relationship between the dry static friction force and the normal force is known once the coefficient of static friction is known, the angle between the normal and the total reaction force at the surface of contact can be determined. The relationship

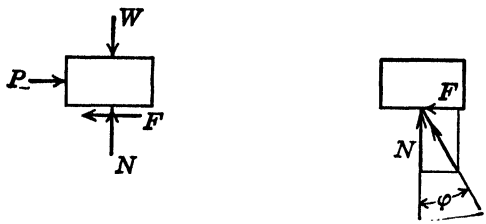


FIG. 5–2

between this angle of friction and the coefficient of friction is shown in Fig. 5–2:

$$\tan \phi = \frac{F}{N} = \mu$$

This angle of friction determines the direction of the resultant reaction force.

There are two types of problems involving dry friction which commonly occur in statics. In the first type of problem, one of the conditions of the physical problem is that motion is impending, and the relationship between the forces is required. For such problems it is known that the maximum possible static frictional forces exist, so that the relationship between the frictional forces and the normal forces is directly given by the coefficient of static friction. In the other type of problem the forces acting on the system are given, and it is desired to know whether the frictional forces can maintain equilibrium of the system. Since in this case frictional forces less than the maximum may suffice to maintain equilibrium, one does not know that the friction forces are given by the normal force times the coefficient of static friction. In this case the usual approach is to determine the coefficient of static friction necessary to maintain equilibrium. By then comparing this calculated coefficient with the coefficient which is possible for the given problem, one can tell whether or not equilibrium will be possible.

EXAMPLE 1. A block of weight W rests on an inclined plane (Fig. 5–3). The coefficient of static friction between the block and

the plane is μ. Find the angle α for impending motion down the plane.

Solution. Drawing a free-body diagram (Fig. 5–4), and taking

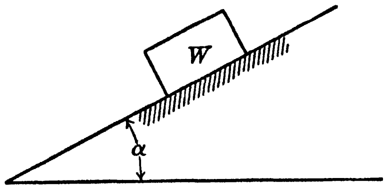


FIG. 5–3

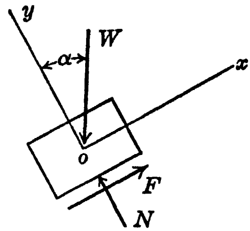


FIG. 5–4

the coordinate system parallel and normal to the plane, we have for equilibrium:

$$\Sigma F_x = 0 = F - W \sin \alpha$$
$$\Sigma F_y = 0 = N - W \cos \alpha$$

Since motion is impending

$$F = \mu N$$

so

$$\frac{F}{N} = \tan \alpha = \mu$$

or, since $\mu = \tan \phi$ where $\phi =$ the angle of friction

$$\alpha = \phi$$

The angle α is called the *angle of repose.*

EXAMPLE 2. A load P is carried on a bracket which slides on a vertical member as shown in Fig. 5–5. At what distance x from the support should the load be placed in order that motion shall be impending? Neglect the weight of the bracket, and assume that there is dry friction between the bracket and the support with a coefficient of static friction equal to 0.25.

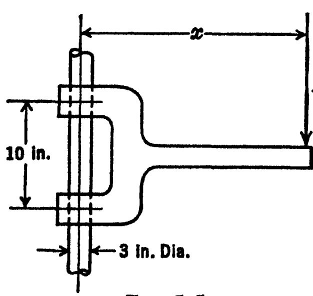


FIG. 5–5

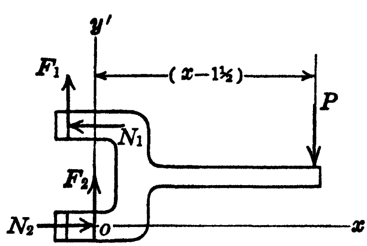


FIG. 5–6

Solution. Drawing a free-body diagram (Fig. 5–6) and writing the equations of equilibrium, we have:

$$\Sigma F_x = 0 = -N_1 + N_2; \quad N_1 = N_2$$
$$\Sigma F_y = 0 = F_1 + F_2 - P$$
$$\Sigma M_0 = 0 = (10)(N_1) - (3)(F_1) - (x - 1.5)(P)$$

Since motion is impending, we also have:

$$F_1 = 0.25N_1$$
$$F_2 = 0.25N_2$$

Hence:

$$P = F_1 + F_2 = 0.5N_1$$
$$10N_1 - 0.75N_1 - (x - 1.5)(0.5N_1) = 0$$
$$10 - 0.75 + 0.75 = 0.5x$$
$$x = 20 \text{ in.}$$

By using the idea of the angle of friction, a graphical solution of this problem can be obtained. If the reaction forces are shown

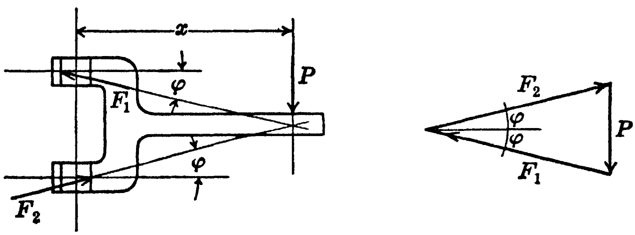


FIG. 5–7

in their proper directions, instead of as two components, the force system will consist of three non-parallel coplanar forces, which must be concurrent if they are to be in equilibrium (Fig. 5–7).

EXAMPLE 3. The block W_1 weighs 100 lb, and the block W_2 weighs 50 lb (Fig. 5–8). The coefficient of static friction between the upper block and the lower block is 0.35, and between the lower block and the horizontal plane is 0.1. A horizontal force of 20 lb acts upon the upper block as shown in the diagram. Find the force in the cable AB for equilibrium of the lower block.

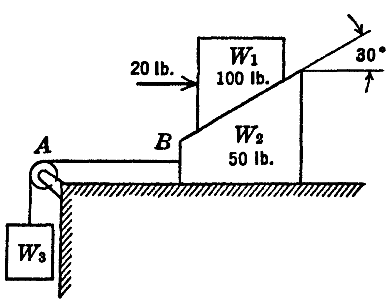


FIG. 5–8

Solution. Since we do not know that the top block is in equilibrium or that motion of this top block is impending, we cannot immediately write down the equations of equilibrium. We must first check to see whether equilibrium is possible, or whether the upper block will move. Let us compute the coefficient of static friction which would just suffice to keep the upper block in equilibrium, i.e., which corresponds to impending motion. It will be seen that if there were no friction force, the upper block would move down the lower block, so that we would have impending motion downward.

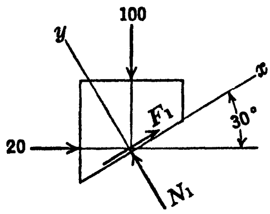


Fig. 5-9

$$\Sigma F_x = 0 = 20 \cos 30^\circ - 100 \sin 30^\circ + F_1$$
$$\Sigma F_y = 0 = -20 \sin 30^\circ - 100 \cos 30^\circ + N_1$$
$$F_1 = \mu N_1$$
$$F_1 = -(20)(0.866) + (100)(0.5) = 32.7 \text{ lb}$$
$$N = (100)(0.866) + (20)(0.5) = 96.6 \text{ lb}$$
$$\mu = \frac{32.7}{96.6} = 0.338$$

Since this friction required for equilibrium is less than that actually present, we know that equilibrium exists. Furthermore, motion is not impending, and the above value of computed friction force (32.7 lb) is the actual value existing, although it is not the maximum value which could exist.

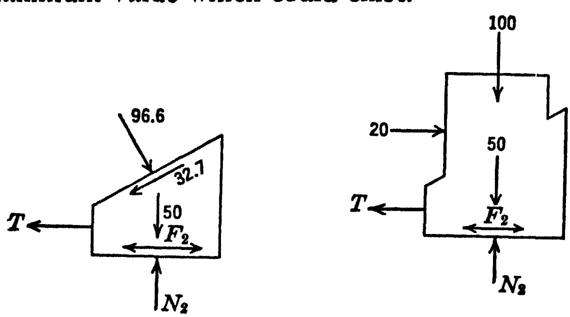


Fig. 5–10

Similarly for analysis of the lower block we have the free-body diagrams shown in Fig. 5–10. Motion of the lower block might

be impending in either direction, and still the system would be in equilibrium. There will thus be a range of values of the force T which could maintain equilibrium.

Assuming motion impending to the left, we have, considering the free-body diagram of the whole body:

$$\Sigma F_x = 0 = -T + F_2 + 20$$
$$\Sigma F_y = 0 = N_2 - 50 - 100; \quad N_2 = 150 \text{ lb}$$
$$F_2 = 0.1N_2 = 15 \text{ lb}$$
$$T = 15 + 20 = 35 \text{ lb}$$

Assuming motion impending to the right, we have:

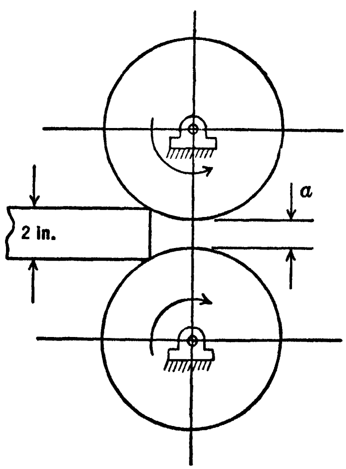
$$\Sigma F_x = 0 = -T - F_2 + 20$$
$$T = 20 - F_2 = 20 - 15$$
$$T = 5 \text{ lb}$$

Hence any value of T between 5 and 35 lb will keep the system in equilibrium. For intermediate values, motion will not be impending, and the magnitude of the friction force will not be its maximum possible value, but will be that required for equilibrium.

PROBLEMS

118. A ladder weighing 75 lb rests against a vertical wall at an angle of 30° with the wall. If the coefficient of static friction between the ladder and the wall and floor is 0.25, how far up the ladder can a 160-lb man climb without slipping of the ladder?

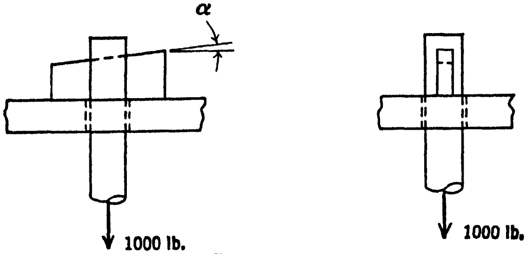
119. A 2-in. thick slab is to be rolled between two 24-in. diameter rolls. If the coefficient of kinetic friction between the rolls and the hot metal is 0.15, what is the minimum spacing a between the rolls for which the forces exerted on the slab by the rolls at first contact will be in such a direction as to pull the slab into the rolls?



PROB. 119

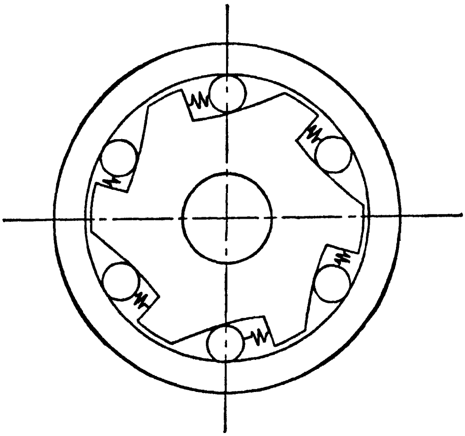
120. A round bar loaded with a 1000-lb weight passes through a plate and is supported by means of a "cotter key" through a slot in the round bar. Assuming that the coefficient of static friction between the key and the bar, between the key and the plate, and between the bar and the plate, is 0.25, find the maximum angle α for

which the device is self-locking, i.e., for which the load does not force the key out of the slot.



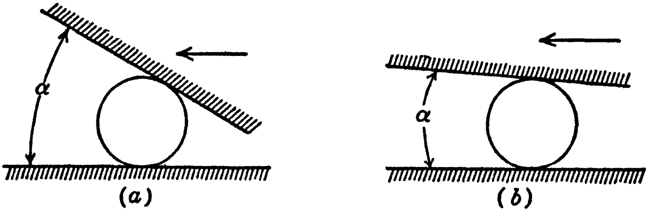
PROB. 120

121. If the angle α of the key in Problem 120 above is 12°, find the force necessary to withdraw the key from the slot.



PROB. 122

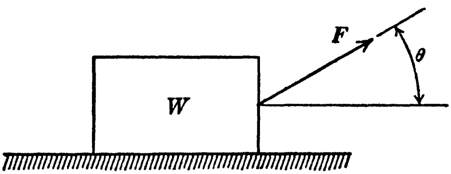
122. One type of free wheeling clutch is shown in the figure. If the inside member is rotated counterclockwise, the rollers are jammed between the inclined surfaces and hence drive the outer member. If the inner member is rotated clockwise, the rollers are pulled free and do not drive the outer member. The situation for each roller is somewhat as shown in the following diagram.



PROB. 122

If the angle between the two surfaces is too great, as in (a), the roller would be just pushed along the lower surface. If the angle is too small, as in (b), the roller will jam itself between the two surfaces and will be difficult to release. If the coefficient of static friction for the hardened steel rollers and surfaces is 0.08, find the maximum angle α which could be used for such a clutch.

123. A block of weight W rests on a rough horizontal plane and is loaded by a force F making an angle θ with the horizontal. Show that

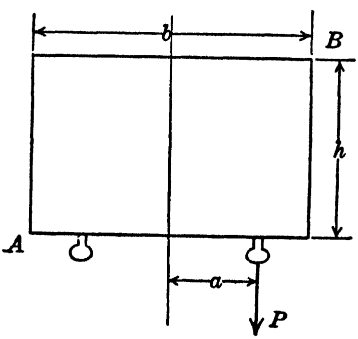


PROB. 123

the minimum value of F is required for impending motion when the angle θ is equal to the friction angle.

124. An insect climbs up the side of a hemispherical bowl. If the coefficient of static friction between the legs of the insect and the bowl is 0.30, how high can the insect climb?

125. A force P is applied to one handle of a drawer. Assuming that the reactions are concentrated at the points A and B, find the maximum distance a at which the drawer could be pulled open, for a given coefficient of static friction μ. Neglect friction on the bottom of the drawer.

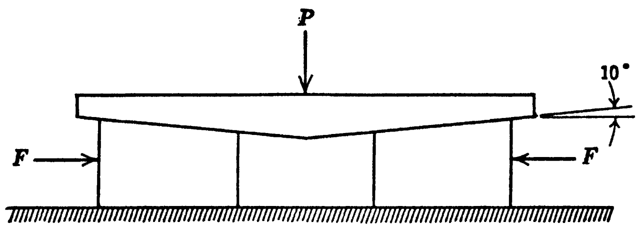


PROB. 125

126. A uniform rectangular block of dimensions 4 in. by 4 in. by 10 in. rests on a horizontal board, with the large dimension normal to the board. The coefficient of static friction between the block and the board is 0.25. If the board is gradually tilted through an increasing angle, will the block slide or tip?

127. A load P is to be raised by a system of wedges loaded by forces F as shown in the diagram. If the coefficient of static friction is 0.25 for all surfaces in contact, find the relationship between F and P to raise

the load and to lower the load. Is the system self-locking; i.e., if the forces F are removed, is P supported?



PROB. 127

Thick-Film and Boundary Lubrication. Dry friction, of the kind discussed above, is most usually found in those machines in which the frictional force is a useful element, such as brakes and clutches, since in these cases it is desirable to have large coefficients of friction. In another large class of applications, however, it is desired to have as low a coefficient of friction as possible, since the friction is an undesirable element and represents a loss of power in the machine. For such applications one uses thick-film lubrication or a rolling action to reduce the frictional forces. The friction forces involved in these types of support are entirely different from the dry friction problem discussed above.

In thick-film lubrication the two contact surfaces are separated by a film of lubricant, such as oil, which is thick enough so that there is no metal-to-metal contact. In this case, the tangential forces necessary to cause motion are only those required to shear the layers of oil, and these frictional forces then depend upon the properties of the lubricating substance rather than on the material or the surface finish of the metals.

Consider two surfaces which are separated by a volume of fluid, as in Fig. 5–11. If we imagine that the oil film is maintained when the surfaces are stationary, then it will be seen that the coefficient of friction for such a situation would be zero, since a fluid at rest by definition is not able to support any tangential forces. In this event any small force F would suffice to set the upper body into motion. The coefficient of kinetic friction, however, is not zero, since it requires a definite force to shear a fluid at a given rate. The magnitude of this force is related to the velocity and the

areas involved and also depends on the viscosity of the fluid, which we now proceed to define. Consider a horizontal layer of

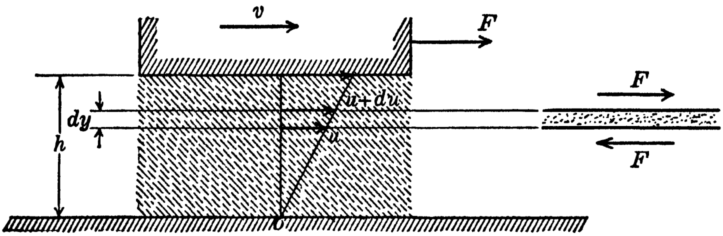


FIG. 5-11

the fluid, of thickness dy, as shown in Fig. 5-11. The lower surface of the layer moves with a velocity u and the upper surface moves with a velocity $u + du$. To maintain this steady flow requires a resultant force F on both surfaces, as shown in the free-body diagram. It was assumed by Newton that this drag force F was directly proportional to the surface area A of the layer, and the velocity gradient du/dy:

$$\mathrm{F} = \mu A \frac{du}{dy}$$

The validity of this assumption is checked by experiment, and it is found for a given fluid at a given temperature that μ is a constant.

In Fig. 5-11 we may find the force F necessary to move the upper surface with a velocity v with respect to the lower surface, where h is the total fluid thickness. If we assume that the fluid elements directly adjacent to the moving surface have a velocity v while the fluid elements directly adjacent to the stationary surface are at rest, and that the velocity distribution across the fluid layer is linear, we have

$$\frac{du}{dy} = \frac{v}{h}$$

and

$$F = \frac{\mu A}{h} v$$

Thus, for thick-film lubrication the total frictional force is directly proportional to the velocity of the body. There is thus seen to be

a great difference between thick-film friction and dry friction, for which the frictional force decreases with velocity (Fig. 5–12).

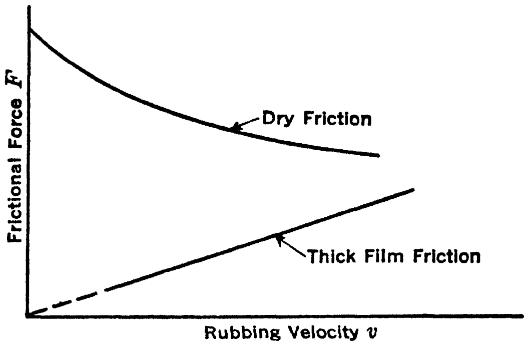


FIG. 5–12

One of the most common applications of thick-film lubrication is to cylindrical bearings. In Fig. 5–13 a shaft of diameter d rotates at n revolutions per minute in a stationary bearing. The thickness of the lubricant between the shaft and the stationary housing is, of course, greatly exaggerated in the diagram. A load P acts upon the shaft. We assume that the bearing is centered in the shaft so that the film of oil is of constant thickness c, and that the velocity varies linearly across the film. Then, if F is the friction force, and l is the length of the bearing:

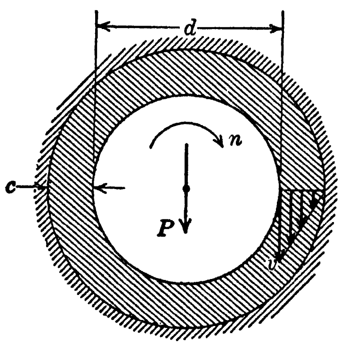


FIG. 5–13

$$F = \mu\pi dl\frac{v}{c} = \mu\pi dl\,\frac{n\pi d}{c}$$

If we define the bearing pressure as the load divided by the projected area of the bearing:

$$p = \frac{P}{dl}$$

we have:

$$F = \mu\pi\frac{P}{p}\frac{n\pi d}{c}$$

The "coefficient of friction" of the bearing can be defined as the total friction force divided by the load

$$\frac{F}{P} = \pi^2 \left(\frac{\mu n}{p}\right)\left(\frac{d}{c}\right)$$

This equation thus relates all of the significant quantities involved in the design of such a bearing.

In an actual bearing, the assumption of an oil film of uniform thickness would not be true. The complete solution of the problem for a non-uniform oil film is a problem in hydrodynamics which is taken up in connection with the theory of journal bearings. The foregoing equation is accurate enough, however, to give a useful first approximation to bearing performance, and it is often used in preliminary design.

There is a third type of friction which is frequently encountered. In many cases it is not possible to maintain a perfect oil film so that there does occur a certain amount of contact between surfaces, or so that the oil film is so thin that the amount of frictional force is affected by attractive forces set up between the surfaces. This so-called *boundary lubrication* will thus depend not only on the lubricant properties, but also upon the surface finishes and materials. There is the additional complication that very thin films of lubricants possess special properties, and such phenomena as an adsorption of the lubricant on the metal surfaces occur. The behavior of this boundary friction is somewhere between that of dry friction and that of thick-film friction. Because of the many factors involved, the coefficients of friction for this case must be obtained experimentally under the exact operating conditions.

Rolling Friction. When two bodies roll together, for example when a wheel rolls on a rail or a roller in a bearing rolls against its race, a somewhat different type of frictional force is set up.

Imagine a wheel loaded by a force W moving to the left along a track with a uniform velocity, as in Fig. 5–14. Because of the continuous deformation of the track, a horizontal force F is required to keep the system in equilibrium, and this force is thus a measure of the rolling resistance of the wheel. The amount of rolling resistance is usually specified by giving the distance a,

which is sometimes called the coefficient of rolling resistance. It should be noted, however, that this distance a should not properly

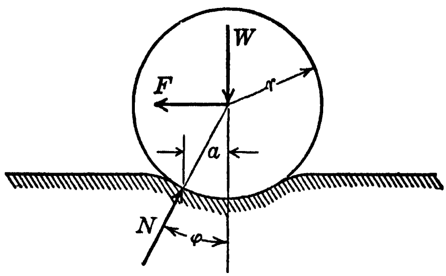


Fig. 5–14

be called a coefficient, and that it is related to the frictional force as follows:

$$F = N \sin \phi$$
$$W = N \cos \phi$$
$$F = W \tan \phi$$

For small values of a, $\tan \phi \approx \frac{a}{r}$

so:

$$F = \frac{Wa}{r}$$

Some representative values of a are given in the following table:

Materials	a (*inches*)
Hardened steel on hardened steel, clean smooth surfaces.......	0.0002–0.0005
Steel on steel..	0.002
Hard wood on hard wood......................................	0.02
Pneumatic rubber tires on smooth road.......................	0.02–0.03

It will be seen that rolling resistances are in general very much smaller than sliding resistance. This fact is made use of in many

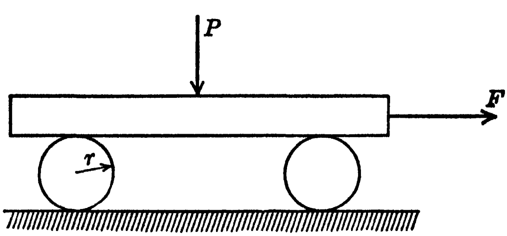


Fig. 5–15

types of so-called "anti-friction" ball and roller bearings. While the coefficients of such bearings are very much less than the coefficient of friction for a bearing with dry or boundary lubrication, they are not less than those possible by means of a correctly designed thick-film lubricated bearing.

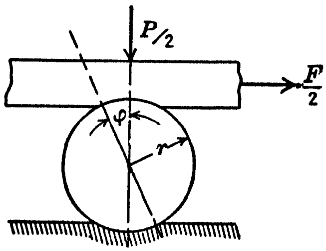


FIG. 5–16

EXAMPLE. A board carrying a load P at its midpoint rests on two cylindrical rollers of radius r, as shown in Fig. 5–15. Find the force F required to move the board.

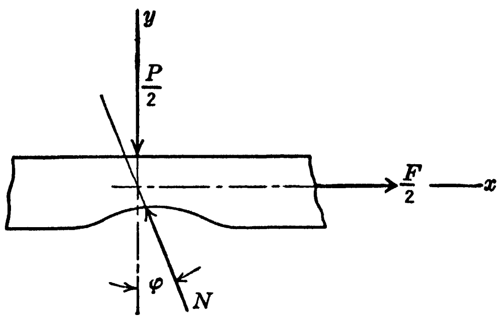


FIG. 5–17

Solution. For one roller the situation is as shown in Fig. 5–16. Drawing a free-body diagram (Fig. 5–17) of the board, we have:

$$\Sigma F_x = 0 = -N \sin \phi + \frac{F}{2}$$

$$\Sigma F_y = 0 = N \cos \phi - \frac{P}{2}$$

hence:

$$N = \frac{P}{2 \cos \phi}$$

$$F = 2N \sin \phi = 2 \sin \phi \left(\frac{P}{2 \cos \phi}\right)$$

$$F = P \tan \phi$$

For small values of ϕ we have $\tan \phi \approx \sin \phi$, also $\sin \phi = \dfrac{a}{r}$,

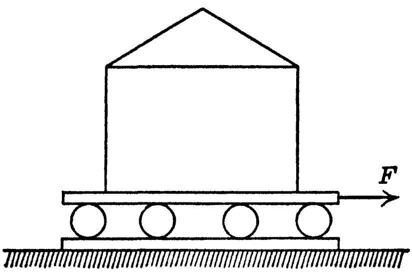
where a is the "coefficient of rolling resistance," as defined above; thus

$$F = \frac{Pa}{r}$$

Note that, although we have in this problem two rolling surfaces, i.e., *double-rolling*, the same answer is obtained that was found above for single-rolling contact. There are twice the rolling surfaces, but the force is applied in such a position that it is twice as effective in producing rolling motion, so that the net result is the same.

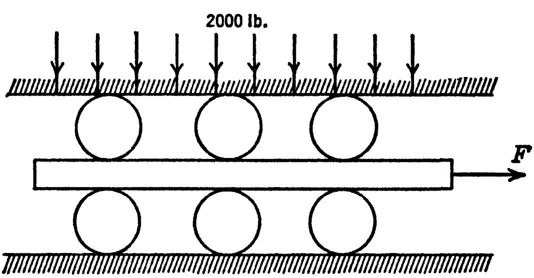
PROBLEMS

128. A house is being moved by pulling it along on planks and wooden rollers as is shown in the diagram. If the total weight of the house is 30,000 lb, find the required horizontal force to move the house at a uniform velocity. The diameter of the rollers is one foot.



PROB. 128

129. A straight board is loaded between two sets of rollers as is shown in the accompanying figure. The diameter of the rollers is 3 in., and the coefficient of rolling resistance for all surfaces in contact is 0.01 in. The whole system is loaded vertically by a uniformly distributed load of 2000 lb. Both the top and bottom surfaces are



PROB. 129

arranged so that they cannot move horizontally. What horizontal force F is required to move the board?

Friction Problems in Machine Design. In the follcwing sections we shall consider a number of cases in which the foregoing

theory of friction can be applied to the design of machine elements. In all cases we shall consider that the friction is of a type which can be expressed by a constant coefficient of friction. Although these conditions are never exactly realized in practice, the conclusions based on the assumption of a constant μ are of value for preliminary design purposes.

Pivot Friction. Given a circular shaft of radius r which is loaded by an axial load of P lb and is supported by a plane surface as shown in Fig. 5–18. Find the friction torque set up at the surface.

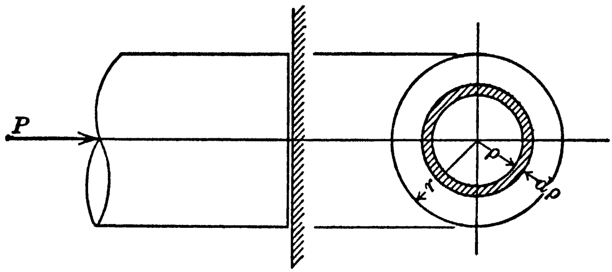


FIG. 5–18

Consider an infinitesimal element in the form of an annular ring of radius ρ and thickness $d\rho$. We must now assume some force distribution over the surface. The simplest assumption will be that of a uniform force distribution. Let us say that the unit force or pressure at any point on the surface is p lb per sq in. This is related to the total force P by the fact that:

$$pA = P = p\pi r^2$$

The normal force acting on the infinitesimal annular area dA is equal to:

$$p\,dA = (p)(2\pi\rho\,d\rho)$$

The frictional force tangent to the surface is:

$$df = \mu p 2\pi\rho\,d\rho$$

The frictional moment about the axis of the shaft, due to the one element shown, is:

$$dM = \rho(\mu p 2\pi\rho\,d\rho)$$

To find the total friction torque we sum up the elements for the whole area:

$$M = 2\pi\mu p \int_0^r \rho^2 \, d\rho = 2\pi\mu p \left(\frac{\rho^3}{3}\right)\Big|_0^r$$
$$= 2\pi\mu \frac{P}{\pi r^2} \frac{r^3}{3} = \frac{2}{3} r\mu P$$

hence, the friction force acts as though it were concentrated at a distance of $\frac{2}{3}r$ from the center.

If there is relative motion between the shaft and the support surface, for example in a thrust bearing, the above assumption of a constant pressure, which is an approximation in any event, is no longer reasonable. Since the velocity is greater at the outside of the shaft, the friction work done by the elements at large values of ρ is greater than the friction work done by the elements nearer the center. This causes the outside elements to wear more than the inside elements, and the pressure becomes non-uniform. Let us next solve for the friction torque under the assumption that it is the wear or friction work which is a constant instead of the pressure. We thus make the quantity $p\rho$ a constant, rather than p alone. If we call $p\rho = K$ then the above equation for the friction torque becomes:

$$M = 2\pi\mu K \int_0^r \rho \, d\rho = 2\pi\mu K \left(\frac{\rho^2}{2}\right)\Big|_0^r = \pi\mu K r^2$$

To find the value of K we note that the sum of all of the normal forces on the elements must be P:

$$P = \int_0^r p 2\pi\rho \, d\rho = 2\pi K \int_0^r d\rho = 2\pi K r$$

so:

$$K = \frac{P}{2\pi r}$$

thus:

$$M = \pi\mu r^2 \frac{P}{2\pi r} = \frac{r}{2} \mu P$$

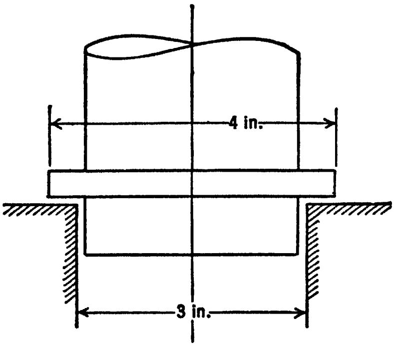
hence, under the assumption of constant wear, the friction force acts as though it were concentrated at a distance of $\frac{r}{2}$ from the center.

In most practical problems the coefficient of friction will not be

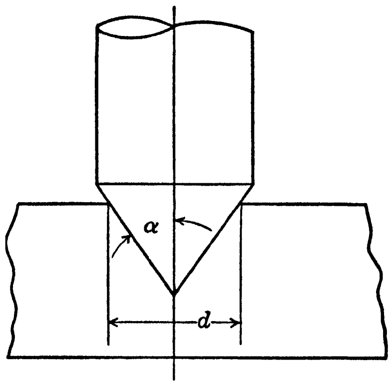
known with sufficient accuracy to justify a distinction between the above two cases.

PROBLEMS

130. A step bearing has the dimensions shown in the diagram. In calculating the energy loss due to friction, how much difference is there



PROB. 130



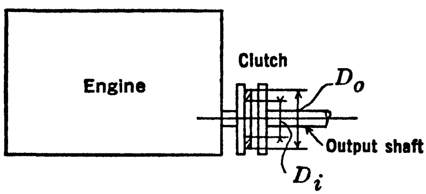
PROB. 131

between the conclusions based on constant wear and on constant pressure between the bearing surfaces?

131. Find the friction torque for a conical pivot point as shown in the diagram, assuming that the pressures are uniformly distributed over the surface. Show that for the angle $\alpha = 90°$ the result reduces to the case of a plane flat pivot as developed above.

132. A certain automobile engine is capable of developing a maximum torque of 2500 in.-lb. A single plate clutch having the form shown in the diagram is to be designed for the engine.

The torque is transmitted by means of the annular friction surface having an outer diameter D_o and an inner diameter D_i. The friction disk has an asbestos facing for which the coefficient of friction is 0.3. If the force per unit area of the disk (assumed to be constant) is to be limited to 30 lb per sq in., and if $D_i = 6$ in., what should be the value of D_o and what axial force is required to engage the clutch?



PROB. 132

Screw Threads. A commonly used machine element in which friction plays an important part is the screw thread. Such devices

are used for various fastenings, for transferring motion, and for exerting large forces. We will consider a screw-jack, as shown in Fig. 5–19a.

We first consider the case in which the axial load P is raised by means of the applied moment $M = Qa$. In Fig. 5–19b, the

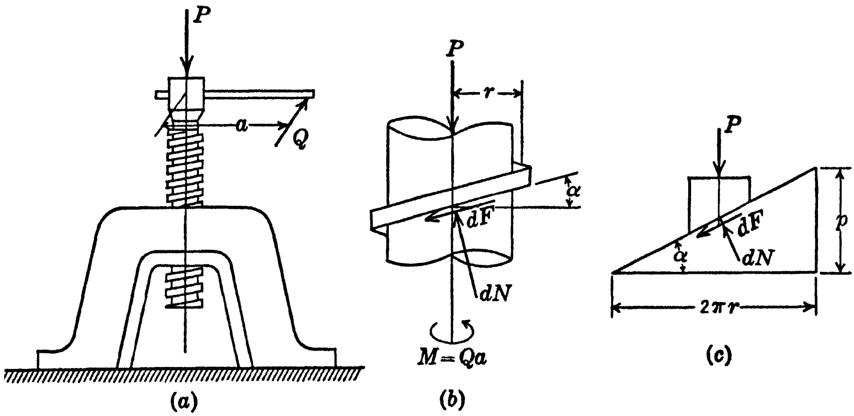


FIG. 5–19

forces acting on the screw thread are shown. In reality, these forces exerted on the screw thread by the nut are distributed all around the thread and usually over several threads. We show in the diagram only the forces acting on one typical element of the nut. We also limit our treatment to a square thread; thus the forces all lie in planes parallel to the axis of the screw. In Fig. 5–19c, the fact that a screw thread is simply an inclined plane wrapped around a cylinder may be seen. The mean radius of the thread, at which the forces are assumed to act, is r and the distance the thread advances in one revolution—the *pitch* of the screw—is p.

Then from the free-body diagram of Fig. 5–19b:

$$\Sigma F_y = 0 = -P + \Sigma(-dF \sin\alpha + dN \cos\alpha)$$

$$\Sigma M = 0 = M - r\Sigma(dF \cos\alpha + dN \sin\alpha)$$

$$M = r\Sigma(dF \cos\alpha + dN \sin\alpha)$$

$$= r(\mu \cos\alpha + \sin\alpha)\Sigma\, dN$$

$$P = (\cos\alpha - \mu \sin\alpha)\Sigma\, dN$$

$$M = Pr\frac{(\mu \cos\alpha + \sin\alpha)}{(\cos\alpha - \mu \sin\alpha)}$$

Dividing through by $\cos \alpha$ and putting $\mu = \tan \phi$, this becomes

$$M = Pr\left(\frac{\tan \phi + \tan \alpha}{1 - \tan \phi \tan \alpha}\right) = Pr \tan (\phi + \alpha)$$

If the load P is to be lowered instead of raised, the direction of the friction force changes, and the expression for the moment becomes

$$M = Pr \tan (\phi - \alpha)$$

It is desirable to design a screw-jack so that it will be self-locking. By this we mean that, if we are raising a load and then stop applying the force, we do not want to have to hold the lever, i.e., to apply a moment, to prevent the load from lowering. The condition for self-locking, therefore, is that the moment required to lower the load must remain positive. From the foregoing formula for the moment required to lower the load, it will be seen that if the pitch angle α is less than the friction angle ϕ the screw will be self-locking.

A screw which is not self-locking is said to be an "overhauling" screw. An overhauling screw and nut system can be driven in either direction; i.e., if the nut is translated, the screw will rotate.

If a screw is to be used as a driving element, the power lost in the machine because of friction may become a relatively large figure, and the efficiency of the screw becomes of significance.

The efficiency of a machine is defined as the ratio of the output to the input. For the problem of the screw-jack discussed above, the input can be computed from the torque required to raise the load, and the output from the magnitude of the load raised. If we suppose that the screw is being rotated at ω rad/sec, then the power input is equal to $M\omega$ ft-lb/sec, if M is the applied moment in ft-lb. The velocity with which the load is raised, if the screw is rotated at ω rad/sec is (see Fig. 5–19c):

$$v = \frac{p \text{ ft/rev}}{2\pi \text{ rad/rev}} \cdot \omega \frac{\text{rad}}{\text{sec}} = \frac{p\omega}{2\pi} \frac{\text{ft}}{\text{sec}}$$

$$v = r\omega \tan \alpha$$

hence the power output is:

$$Pr\omega \tan \alpha$$

and

$$\text{efficiency} = \frac{\text{output}}{\text{input}} = \frac{Pr\omega \tan \alpha}{Pr\omega \tan (\alpha + \phi)}$$
$$= \frac{\tan \alpha}{\tan (\alpha + \phi)}$$

This same conclusion can be reached from a different viewpoint by noting that the efficiency can also be defined as the ratio of the torque which would be required to raise the load if there were no friction to the torque which is actually required.

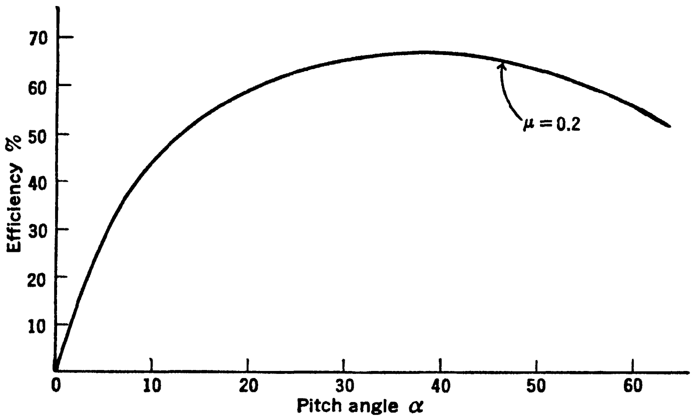


Fig. 5–20

If we plot the efficiency of the screw versus the pitch angle α, we find that the curve has a definite maximum point, as may be seen in Fig. 5–20.

PROBLEMS

133. Show that the moment required to lower a load by means of a screw-jack is $M = Pr \tan (\phi - \alpha)$ as given above. Do this by showing in the free-body diagram the resultant of the normal force and the frictional force acting at the friction angle ϕ rather than by carrying the two components through the analysis as was done above.

134. A load of 10,000 lb is to be lifted by means of a square-threaded screw having two threads per inch and a mean thread diameter of 2 in. If the coefficient of friction is 0.2, what torque is required to raise the load?

135. Show that the maximum efficiency which a self-locking screw can have is 50%.

Brake-Band or Belt Friction. Another type of friction problem frequently occurs in machine design when a flexible member such as a brake-band, cable, or belt passes over or is wrapped around a cylinder. Three typical cases are shown in Fig. 5–21. In (a) a

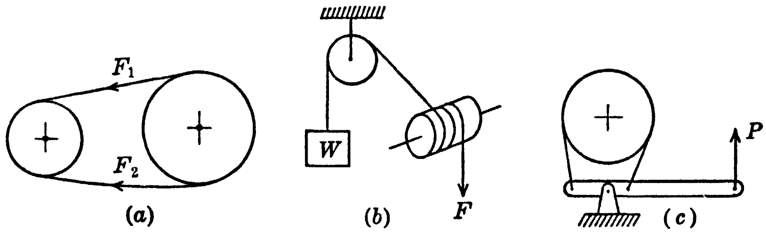


Fig. 5–21

belt is shown passing around two pulleys. If there were no friction between the pulley and the belt, the forces F_1 and F_2 would be equal, and there would be no resultant torque on the pulley and hence no transmitted power. In (b) a weight W is supported by a cable wrapped around a drum. If there were no friction between the cable and the drum, the force F would have to be equal to W for equilibrium. By making use of friction, however, W can be supported with a much smaller force F. In (c) a rotating pulley is braked by means of a circumferential brake-band which is tightened by means of the lever system and the force P. Because of the friction between the brake-band and the pulley, a torque acts on the pulley to slow it down or to keep it at rest.

In the foregoing examples it will be seen that any condition of rest, relative motion, or impending motion may exist. It is

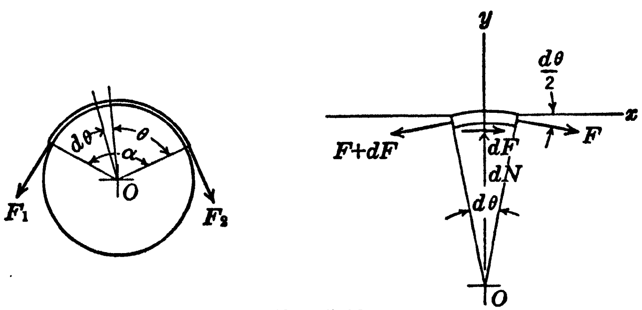


Fig. 5–22

assumed that the coefficient of friction in the following analysis has been taken as the appropriate one for the particular conditions.

In Fig. 5–22 we show a portion of a belt or brake-band, and we assume that relative motion is either present or impending so that we can say $dF = \mu\, dN$, where μ is assumed to be constant. The total arc of contact of the band is α, and $d\theta$ is the infinitesimal angle subtended by the element of the belt shown in the free-body diagram of Fig. 5–22b. Writing the equations of equilibrium for this element, we have:

$$\Sigma F_x = 0 = F \cos\frac{d\theta}{2} - (F + dF)\cos\frac{d\theta}{2} + \mu\, dN$$

$$\Sigma F_y = 0 = dN - (F + dF)\sin\frac{d\theta}{2} - F \sin\frac{d\theta}{2}$$

Putting $\cos\frac{d\theta}{2} \approx 1$, $\sin\frac{d\theta}{2} \approx \frac{d\theta}{2}$ and dropping second-order differentials, this becomes:

$$-dF + \mu\, dN = 0$$

$$dN - F\, d\theta = 0$$

Eliminating dN:

$$\frac{dF}{\mu} = F\, d\theta$$

$$\frac{dF}{F} = \mu\, d\theta$$

Integrating this equation over the whole arc of contact, we obtain:

$$\int_{F_2}^{F_1} \frac{dF}{F} = \mu \int_0^{\alpha} d\theta$$

$$\mu\alpha = \log_e \frac{F_1}{F_2}$$

$$\frac{F_1}{F_2} = e^{\mu\alpha}$$

EXAMPLE. What is the minimum force P required to support the load W by the cable as shown in Fig. 5–23? The coefficient of static friction between the cable and the non-rotating cylinder is 0.3.

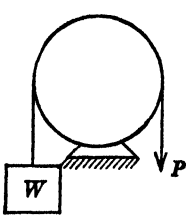


FIG. 5–23

Solution. The minimum force P will be required when downward motion of W is impending; hence the coefficient of static friction

applies and $\mu = 0.3$. The arc of contact α is 180° or π radians so that:

$$\frac{W}{P} = e^{(0.3)\pi} = 2.57$$
$$P = 0.389\,W$$

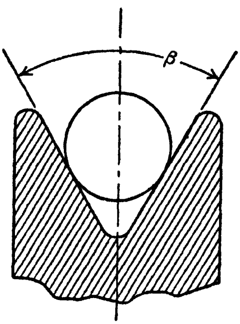
PROBLEMS

136. Referring to Fig. 5–21b, how many times should the cable be wrapped around the cylinder so that a load W of 1000 lb can be supported by a force F of 25 lb? Assume a coefficient of static friction of 0.25.

137. Show that the friction torque M_f of a brake-band, in terms of the maximum force in the band F_1, is given by the following expression:

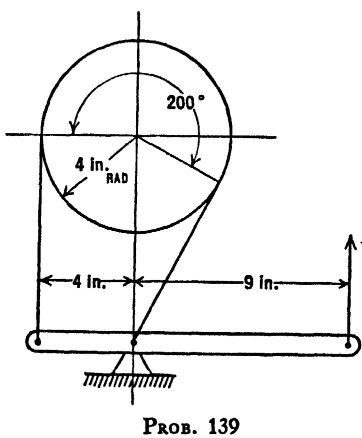
$$M_f = F_1 r\left(\frac{e^{\mu\alpha} - 1}{e^{\mu\alpha}}\right)$$

138. A hoisting pulley is grooved as shown in the figure. Derive the relationship between the tight and loose sides of the cable, and compare with the case of the flat pulley.

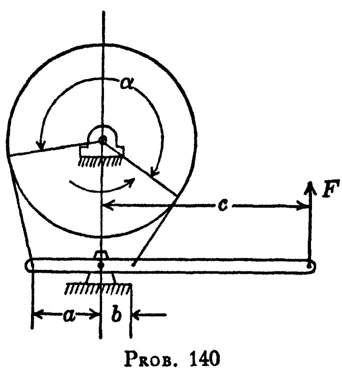


PROB. 138

139. What braking force P is required to exert a friction moment of 1000 in.-lb to prevent clockwise rotation of the drum shown in the figure? The angle of contact is 200° and the coefficient of static friction is 0.2. What would be the required force if the impending motion of the drum were counterclockwise?



PROB. 139



PROB. 140

140. In a differential band-brake, the lever arm is so arranged that the tight side of the band aids the actuating force while the loose side opposes it. Show that such a mechanism can be made so as to permit rotation of the drum in one direction only, while automatically locking in the other direction. Such a device is frequently used on hoists to prevent the load from dropping under its own weight.

CHAPTER VI

ELEMENTARY THEORY OF STATICALLY DETERMINATE STRUCTURES

The use of diagrams is a particular instance of that method of symbols which is so powerful an aid in the advancement of science. A diagram differs from a picture in this respect, that in a diagram no attempt is made to represent those features of the actual material system which are not the special objects of our study . . . when studying the equilibrium of a framework . . . in which each piece acts only by tension or pressure between its extremities, it is not necessary to know whether a particular piece is straight or curved or what may be the form of its section. In order, therefore, to exhibit the structure of the frame in the most elementary manner we may draw it as a skeleton in which the different joints are connected by straight lines. — James Clerk Maxwell, "On Bow's Method of Drawing Diagrams in Graphical Statics" (1876), *Scientific Papers*, Vol. II, p. 492.

Many of the most important applications of the theory of statics are in the field of structural design. One of the first steps in the design of any structure is the calculation of the forces which act in the various members of the structure. Some of the special applications of statics which are used extensively in this field will be discussed in the following pages.

Rigid Frameworks. One very common type of structure consists of straight members so connected together that they form a rigid framework. The members of such a framework are usually long, slender bars loaded in tension or compression. The meaning of the term "rigid framework" may be seen in Fig. 6–1.

In Fig. 6–1a, a framework consisting of four bars pinned together at their ends is shown. It will be seen that such an arrangement is not a rigid framework, since the whole system can deform as shown by the dotted lines, without changing the dimensions of the members. This deformation could be prevented, however, by the addition of one more member as shown in Fig. 6–1b. The framework, which now consists of two triangular sections, will be seen to be rigid. The simplest rigid system is the triangular

framework, Fig. 6–1c, and any complex system which is formed by triangular sections, as Fig. 6–1d, will also be a rigid framework.

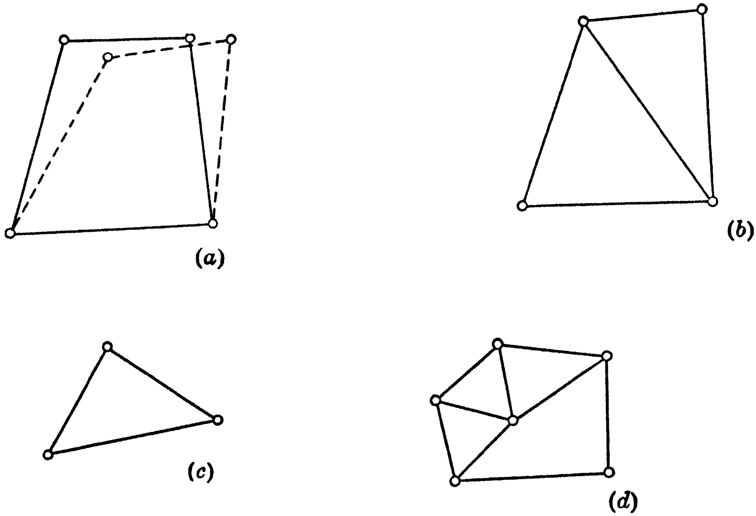


Fig. 6–1

For a rigid framework, there is a definite relationship between the number of members, m, and the number of joints, j. Suppose that one of the members of a planar framework is fixed with respect to a coordinate system. If the whole frame is to be rigid, then the remaining $(j - 2)$ joints must have positions in this coordinate system which are determined by the given lengths of the members. Each of the $(j - 2)$ joints has its two coordinates fixed by the intersection of two additional members. Since one bar is already fixed we have $(m - 1)$ members left to locate the joints; hence:

$$\frac{(m - 1)}{2} = (j - 2)$$

$$m = 2j - 3$$

This relationship is the necessary condition for a rigid planar framework. It is not, however, a sufficient condition, as the members may be so arranged that they do not contribute to the rigidity of the framework. This is illustrated in Fig. 6–2.

In Fig. 6–2a a rigid framework is shown which satisfies the above condition. By moving the member AB to the position BC

in Fig. 6–2b, the frame becomes non-rigid, since the right-hand shaded section could be translated with respect to the left-hand shaded section. Thus the frame in (b) is non-rigid even with the same relationship between members and joints, since the bar *BC* is so placed that it does not maintain the complete constraint of the frame. If the necessary condition is fulfilled, however, one

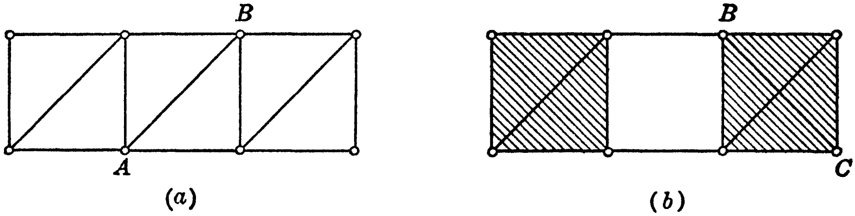


FIG. 6–2

can usually easily tell by inspection whether the frame is rigid, or whether an exceptional condition such as illustrated above is present.

We next consider how to constrain a rigid frame in space by means of supporting members which will exert reactive forces upon the framework when it is loaded by a system of external forces. We shall suppose that the framework is loaded only by forces which lie in the plane of the frame, and hence we need only consider constraint of the frame in a plane.

Following the same line of reasoning as used in Chapter II to establish the condition of complete constraint of a body in space, we find that to constrain a body completely in a plane the equivalent of three non-parallel, non-concurrent forces must be used, as shown in Fig. 6–3a.

A common means of accomplishing this constraint is shown in Fig. 6–3b, where one end of the frame is supported by a pin which can exert a reactive force in any direction in the plane, while the other end is supported in such a way that a vertical reactive force only can be developed. These supports are represented by the three reactive forces A_x, A_y, and B, in the free-body diagram of Fig. 6–3c.

We now suppose that the framework which is constrained by its supports is loaded by a system of external forces (as F_1, F_2, and F_3 in Fig. 6–3), and we investigate the problem of finding the

unknown forces in the members and the unknown reactions at the supports. The total number of unknown elements will be the $(2j - 3)$ axial forces in the members, plus the three reaction forces, so that the total number of unknown forces is $(2j)$. The

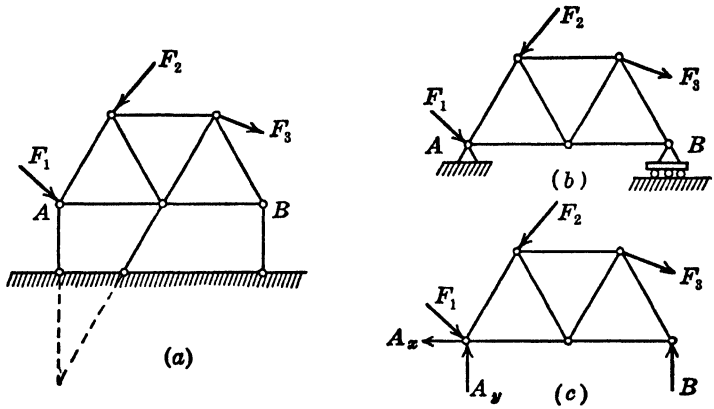


FIG. 6–3

forces acting at each joint of the structure form a concurrent, coplanar force system, for which two independent equations can be written. Thus, the total number of independent equations that can be written for the whole system is $(2j)$, which will just suffice for the determination of the $(2j)$ unknowns. Such a system is said to be *statically determinate*, and the equilibrium equations alone will suffice for the solution of all of the unknown forces in the system.

It is possible, however, that there may be more members in a system than are required for rigidity. If, for example, in Fig. 6–4b



FIG. 6–4

we add the member *AB* to the already rigid frame of Fig. 6–4a, we will have more unknown forces than we have equations.

Such additional members are called *redundant* members, and

their presence makes the problem *statically indeterminate*, since there will not be enough equilibrium equations to solve for the unknown forces. To solve such statically indeterminate problems the deformations of the members must be taken into account, and hence the theory of elasticity must be brought into the problem. We shall confine ourselves, in the present chapter, to statically determinate systems.

Plane Trusses. A *plane truss* is a rigid framework satisfying the following conditions:

(1) The members of the framework all lie in the same plane.
(2) The members are connected at their ends by frictionless pins.
(3) All of the external loads lie in the plane of the framework and are applied at the joints only. This means that the bending effect of the weights of the bars is neglected, since all of the external loads, including the weights of the members, are assumed to act at the joints.

It will be seen that each of the members of a plane truss is acted upon only by forces at its two ends and hence is a two-force member. The lines of action of the forces applied to the members will thus lie along the centerlines of the members.

It will be evident that many actual structures will depart considerably from these conditions. The members, for example, are often welded together at their ends rather than being joined by smooth pins or hinges, so that an additional rigidity is introduced into the structure. In some structures the external loads may be applied at such a location that an appreciable amount of bending occurs in the members. The stresses (force per unit area) in the structure which are due to the axial forces computed on the assumption that the structure may be considered as a plane truss are called the *primary stresses*. Those stresses which are due to departures from the conditions for a plane truss are called *secondary stresses*. In most practical problems the primary stresses are so much larger than the secondary stresses that these secondary stresses can be neglected. In the following paragraphs we shall discuss several methods of determining the axial forces in plane trusses.

The Method of Joints. The method of joints consists of drawing a free-body diagram of each of the joints of the truss and of writing down the two equilibrium equations for each free-body diagram. For most trusses it will be possible to start with a free-body diagram in which there are only two unknowns. The general method will be illustrated by the following example, which will indicate the approach and the steps involved for the analysis of any similar type of truss by the method of joints.

EXAMPLE. A plane truss is acted upon by forces as shown in Fig. 6–5. Find the force in each member of the truss. All of the

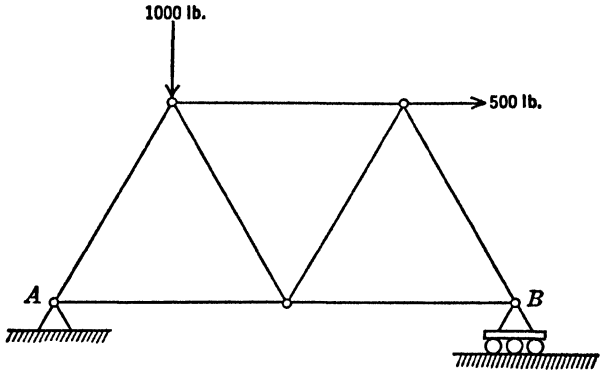


FIG. 6–5

members are of the same length. The left-hand end of the truss is fastened by means of a pin, while the other end is supported on rollers in such a way that the reaction force is vertical.

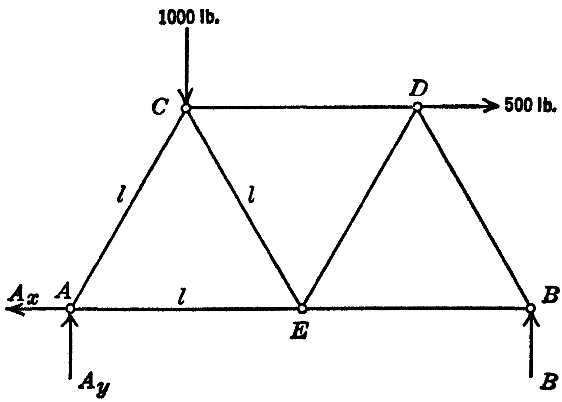


FIG. 6–6

Solution. We first draw a free-body diagram (Fig. 6–6) of the whole structure:

We note that there are no joints which involve only two unknown forces, so that the first step is to determine, from the complete free-body diagram, the external reactions A_x, A_y, and B.

$$\Sigma F_x = 0 = -A_x + 500; \quad A_x = 500 \text{ lb}$$

$$\Sigma M_A = 0 = -\left(\frac{l}{2}\right)(1000) - \left(\frac{\sqrt{3}}{2}l\right)(500) + (2l)B$$

$$B = 250 + 216 = 466 \text{ lb}$$

$$\Sigma M_B = 0 = -(2l)A_y + \left(\frac{3}{2}l\right)(1000) - \left(\frac{\sqrt{3}}{2}l\right)(500)$$

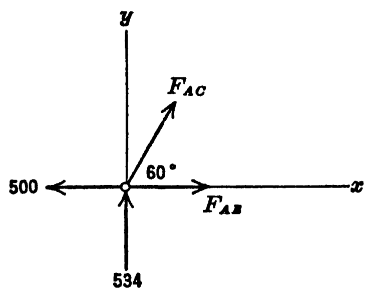
$$A_y = 750 - 216 = 534$$

Check equation:

$$\Sigma F_y = 0 = A_y - 1000 + B$$
$$534 - 1000 + 466 = 0$$

Having found A_x, A_y, and B, we can now start at either joint A or B, since there will be only two unknown forces at each joint.

Joint A:

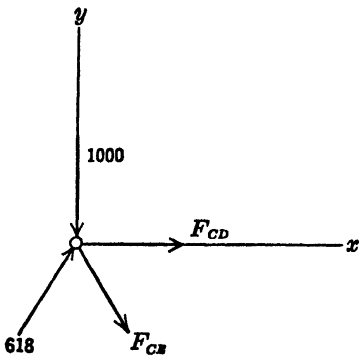


The unknown forces F_{AC} and F_{AB} may be either tensions directed away from the joint, or compressions directed toward the joint. For many trusses it will be possible to tell by inspection whether the forces are tension or compression. If this is not possible, the force can be indicated in either direction. If the force then comes out positive, the assumed direction was correct; if it comes out negative, the force was actually reversed. Another procedure, which we will adopt in the present example, is to indicate all unknown forces as tensions. Then, positive forces are tensions, and negative forces are compressions.

Writing down the equations for the free-body diagram of Joint *A*, we have:

$$\Sigma F_x = 0 = -500 + F_{AE} + F_{AC} \cos 60°$$
$$\Sigma F_y = 0 = +534 + F_{AC} \sin 60°$$
$$F_{AC} = -\frac{534}{(0.866)} = -618 \text{ lb (compression)}$$
$$F_{AE} = 500 - (-618)(\tfrac{1}{2})$$
$$= 500 + 309 = 809 \text{ lb (tension)}$$

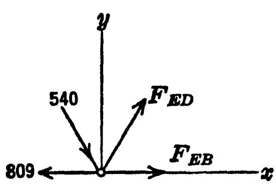
Care must be taken to carry the negative sign on the force F_{AC} through subsequent calculations, as in the $\Sigma F_x = 0$ equation above.



Having found the force F_{AC}, we can now proceed to the free-body diagram of the Joint *C*, since there are only two unknown forces left at that joint. Since we now know that F_{AC} is a compressive force, we shall show it this way in the free-body diagram, rather than carry along the negative sign:

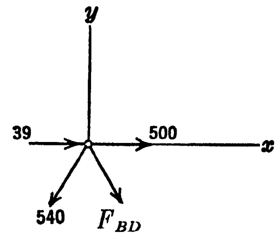
$$\Sigma F_x = 0 = F_{CD} + 618 \cos 60° + F_{CE} \cos 60°$$
$$\Sigma F_y = 0 = -1000 + 618 \sin 60° - F_{CE} \sin 60°$$
$$F_{CE} = \frac{(618)(0.866) - 1000}{0.866} = -540 \text{ lb (compression)}$$
$$F_{CD} = -(618)(\tfrac{1}{2}) - (-540)(\tfrac{1}{2})$$
$$= -309 + 270 = -39 \text{ lb (compression)}$$

Proceeding next to Joint *E*:



$$\Sigma F_x = 0 = F_{EB} + F_{ED} \cos 60° + 540 \cos 60° - 809$$
$$\Sigma F_y = 0 = -540 \sin 60° + F_{ED} \sin 60°$$
$$F_{ED} = 540 \text{ lb (tension)}$$
$$F_{EB} = -(2)(540)(\tfrac{1}{2}) + 809 = 269 \text{ lb (tension)}$$

Joint D:

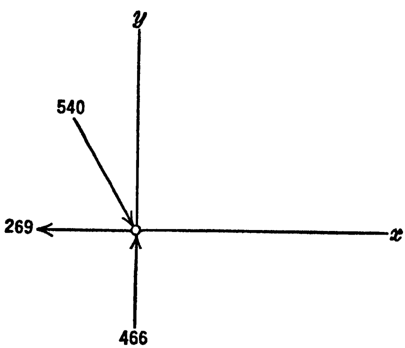


$$\Sigma F_y = 0 = -540 \sin 60° - F_{DB} \sin 60°$$
$$F_{BD} = -540 \text{ lb (compression)}$$

Check equation:

$$\Sigma F_x = 0 = 39 + 500 - 540 \cos 60° - 540 \cos 60°$$
$$539 - 540 \approx 0 \text{ checks, to slide-rule accuracy}$$

We have now found the forces in all of the members, although we have not as yet written the two equations for the Joint B. These two equations will serve as additional check equations.



$$\Sigma F_x = 0 = -269 + 540 \cos 60°; \ -269 + 270 \approx 0$$
$$\Sigma F_y = 0 = 466 - 540 \sin 60°; \ 466 - 466 = 0$$

The Method of Sections. Another method of finding the forces in the members of a truss, the method of sections, is particularly useful for those problems in which the forces in only one or several members of a truss are to be computed. The method consists of drawing a free-body diagram of some portion of the truss, obtained by imagining that the members for which the forces are to be determined have been cut. This will usually result in a general coplanar force system for which three equations of equilibrium can be written.

EXAMPLE. Referring to the truss solved above by the method of joints, find the force in the member CD by the method of sections (Fig. 6–7).

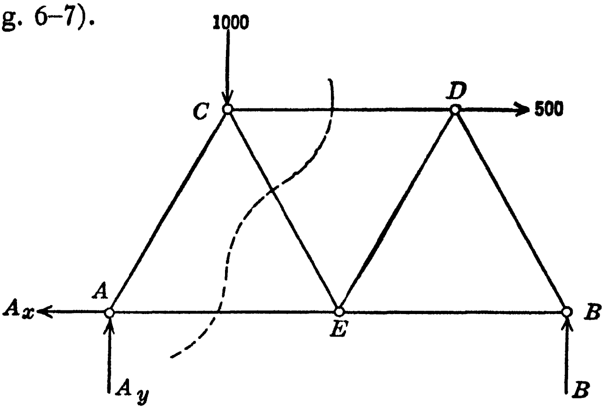


FIG. 6–7

Solution. We imagine that we cut the three bars CD, CE, and AE, and we draw a free-body diagram (Fig. 6–8) of the left-hand portion of the truss:

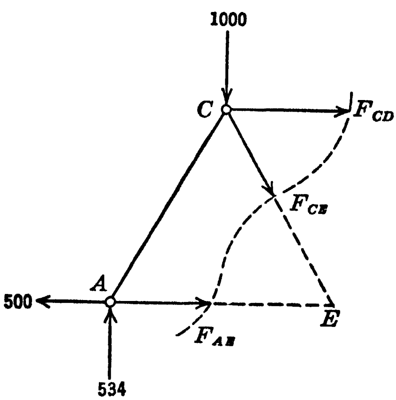


FIG. 6–8

This is a general coplanar force system for which we can write three independent equations. If we determine A_x and A_y as in the previous solution by means of a free-body diagram of the whole truss, there will be left the three unknowns F_{CD}, F_{CE}, and F_{AE}, which can be found from the three equations.

If we require only the force F_{CD}, we can find it directly from one moment equation by taking as a moment center the point E, which is the point of intersection of the two unknown forces F_{CE} and F_{AE}

$$\Sigma M_E = 0 = -(534)(l) - (F_{CD})\left(\frac{\sqrt{3}}{2}l\right) + (1000)\left(\frac{l}{2}\right)$$

$$F_{CD} = \frac{500 - 534}{0.866} = -39 \text{ lb (compression)}$$

which is seen to check the solution by the method of joints.

The same solution could be reached, of course, by taking the right-hand portion of the truss as the free-body diagram (Fig. 6–9):

$$\Sigma M_E = 0 = (466)(l) + (F_{CD})\left(\frac{\sqrt{3}}{2}l\right) - (500)\left(\frac{\sqrt{3}}{2}l\right)$$

$$F_{CD} = \frac{(500)(0.866) - 466}{0.866} = -39 \text{ lb (compression)}$$

In this case the same amount of work is involved for each portion of the truss. In some problems, one portion will lead to simpler equations than the other and should thus be chosen.

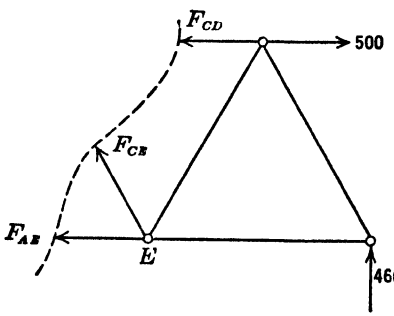


FIG. 6–9

It will be seen that the foregoing method of sections is very generally applicable. Most trusses are so arranged that a portion of the truss can be isolated by cutting three bars, and a moment equation can always be written about the point of intersection of two of the bars. This leaves one equation and one unknown which can be solved directly. The advantage of this method over the method of joints is that the force in a member somewhere in the middle of a truss can be computed without first computing the forces in a number of other members. This advantage is, of course, of less importance if the forces in all of the members of the truss are to be determined. Even in this case it is usually worthwhile to solve for the force in one of the central members as a check on the numerical calculations.

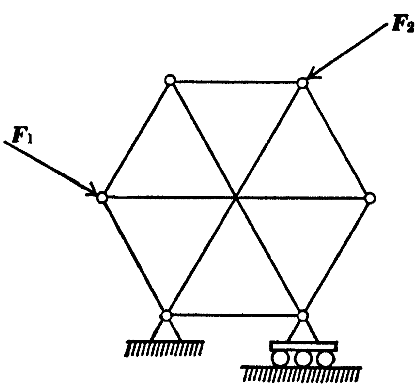


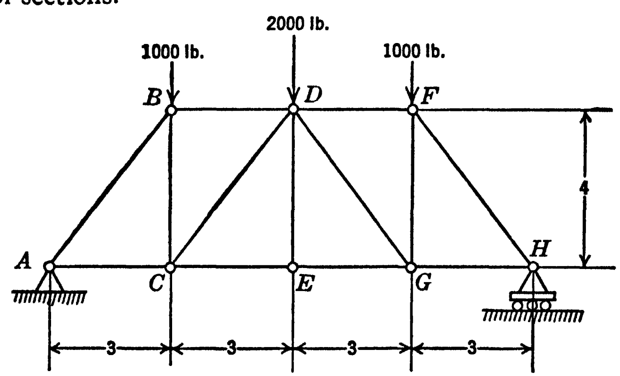
FIG. 6–10

There are certain types of statically determinate trusses which cannot be directly solved by the above methods. Consider, for example, the truss shown in Fig. 6–10.

In this truss, the three members which intersect at the center are not connected there, but freely pass each other at that point. The number of joints in the truss is six, and the number of members is nine, so that the necessary condition for a statically determinate structure is satisfied. It will also be seen that the framework is rigid. If we try to apply the method of joints to this truss, however, we find that there is no joint with just two unknowns, so that we cannot start to unravel the solution. Similarly, with the method of sections, it will be found that the only sections which can be isolated by cutting three bars result in three concurrent unknown forces, for which no solution can be obtained. It is possible, however, to write all of the equations for all of the joints, and then to solve these equations simultaneously. This is, however, a very laborious procedure, which for larger trusses would hardly be a practical mode of calculation. Such trusses are called *complex trusses*, and special methods have been evolved for solving them. There is usually little reason to employ such a complex truss, since a simple truss could in most cases be designed which would be just as effective.

PROBLEMS

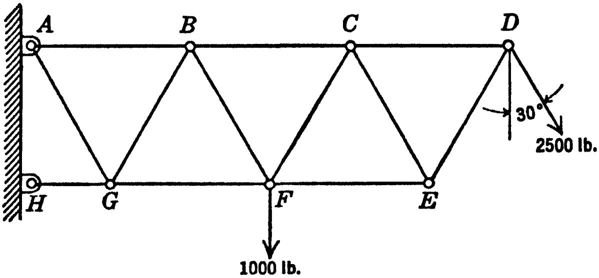
141. Find the forces in the members of the truss shown in the diagram by the method of joints. Check the force in the member *CE* by the method of sections.



PROB. 141

142. All of the members of the truss shown in the diagram have the same length l except *GH*, which has a length $\frac{l}{2}$. Find the force in each

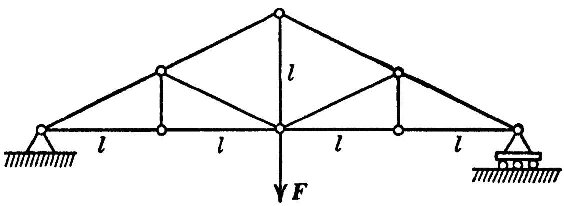
member by the method of joints, and check the force in the member *FG* by the method of sections.



Prob. 142

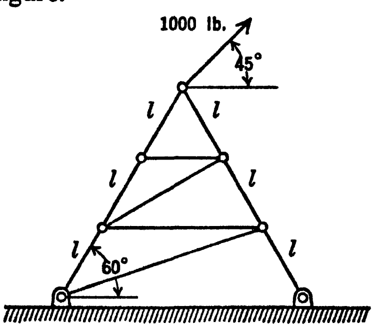
143. (a) Show that if two non-colinear members only meet at a joint, and there is no external load on the joint, the forces in both members must be zero.

(b) Show that if three members meet at a joint with no external load, and two of the members are colinear, the force in the third member is zero.



Prob. 144

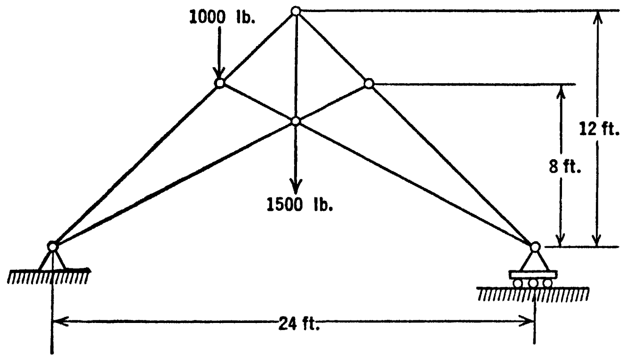
144. Find the force in the central vertical member marked *l* in the truss shown in the figure.



Prob. 145

145. Find the forces in the members of the truss shown in the diagram.

146. Find the force in each member of the truss shown in the figure.



PROB. 146

Graphic Statics. Because of the large amount of routine numerical computation required to solve complicated trusses of many members by the above analytical methods, a number of graphical methods have been developed. These graphical methods have the advantage that the work can be organized as a routine procedure and can thus be carried on, in some instances, by relatively unskilled computers. In addition, some graphical methods are self-checking, so that one can see at a glance if errors have been made.

The methods to be discussed in the following paragraphs have wider application than just the computation of forces in trusses. The methods are also commonly used in connection with the determination of stresses and deflections of beams, and of the critical speeds of rotating machinery.

The Graphical Solution of a Coplanar Force System. The problem of finding the resultant of a coplanar force system can be solved by means of a graphical solution. We shall illustrate the method first by means of a parallel coplanar force system.

Consider the system of three forces shown in Fig. 6–11a. We can find the magnitude and direction of the resultant by adding the forces by the polygon law, as in Fig. 6–11b, but this will not give us the location of the resultant. We can locate the resultant, however, by means of an extension of the graphical method which was developed in connection with Fig. 1–3, Chapter I.

We first resolve the force F_1 into two components, which may have any direction. These components are shown as F_{1a} and F_{1b}

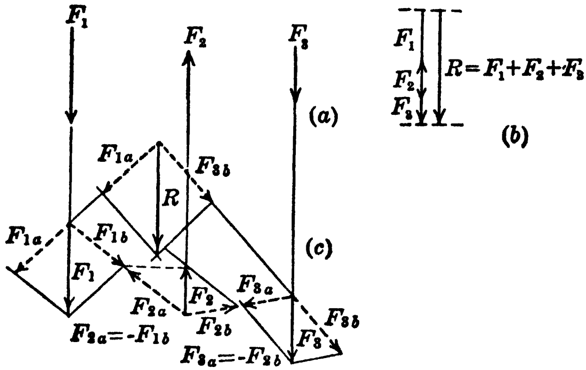


FIG. 6–11

in Fig. 6–11c. We next resolve the force F_2 into two components, one of which is equal in magnitude, opposite in direction, and colinear with the component F_{1b} of the first force. Similarly, the third force is resolved into two components, one of which is equal in magnitude, opposite in direction, and colinear with the component F_{2b} of the second force. We have thus replaced the original system of three parallel forces by a system of six non-parallel forces. Furthermore, four of these forces occur in equal and opposite pairs, so that they cancel out, and we have thus reduced the system of three parallel forces to a system of two non-parallel forces for which the resultant can be directly found. It will be seen that this method could be extended to include any number of forces, and that this system of any number of parallel forces will always be reduced to two forces.

The value of the above method lies in the fact that the various operations required for the solution can be combined into a simple graphical solution. In Fig. 6–12a are shown to scale three parallel forces. To identify the forces we shall use a new type of notation, for reasons that will become apparent as the graphical solution proceeds. Instead of lettering the forces, we letter the spaces between the forces. The first force in Fig. 6–12a would be called the force *ab*, the second, the force *bc*, and so on. This notation is called "Bow's Notation" * and is widely used in the field of

* Bow, R. H., *Economics of Construction in Relation to Framed Structures*, 1873.

graphic statics. In Fig. 6–12b the force diagram is drawn to scale, and the forces are identified by means of the letters of Fig. 6–12a.

The magnitude of the resultant force *ad* can be directly measured from this diagram. To find the location of the resultant, we

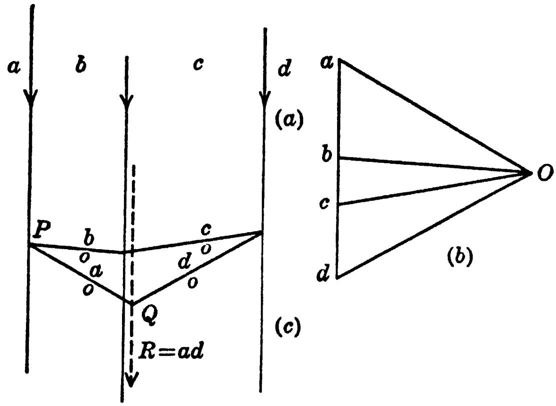


Fig. 6–12

resolve the individual forces into components according to the following scheme. In Fig. 6–12b we pick a point *O* which we call the *pole*. The exact location of the pole is unimportant. We next draw lines from the points *a*, *b*, etc., to the pole. These lines are called *rays*. Since these rays are located in the force diagram, they represent forces. It will be seen, for example, that the rays *ao* and *ob* represent two components of the force *ab*, and the rays *bo* and *oc* are two components of the force *bc*. Furthermore, the components *ob* and *bo* are equal and opposite, so that this construction gives us a mode of making the type of force resolution discussed in connection with Fig. 6–11 above.

We next transfer these force components to the diagram of Fig. 6–12c. Since each force is to be resolved into two components, some point on the line of action of each force will be intersected by two lines parallel to the rays which represent the two force components. We can begin the diagram by selecting any point *P* on the line of action of the force *ab*, and by drawing through this point lines parallel to the rays *ao* and *bo*. Having drawn the line *bo* in Fig. 6–12c, the intersection of *bo* with the line of action of the force *bc* can be found, and through this point a line parallel to

the ray *co* can be drawn. Each of the original forces is resolved into two components, all of which except two cancel out. The remaining two components *ao* and *od* then give the resultant force *ad*, and their point of intersection *Q* in Fig. 6–12c establishes the location of the resultant. Since we have already determined the magnitude of *R* from the force diagram of Fig. 6–12b, we do not need to show the force magnitudes in Fig. 6–12c which we use only to determine the location of the resultant.

The figure formed by the lines *ao*, *bo*, *co*, and *do*, in Fig. 6–12c is called the *funicular* or *string* diagram, or polygon, since it has the form which would be assumed by a flexible string loaded by the original forces.* The lines in the funicular diagram which are parallel to the rays of the force diagram are called *strings*.

In the above example, we have taken a system which has a single force as a resultant. The method is also applicable if the resultant is a couple, as may be seen from the example of Fig. 6–13.

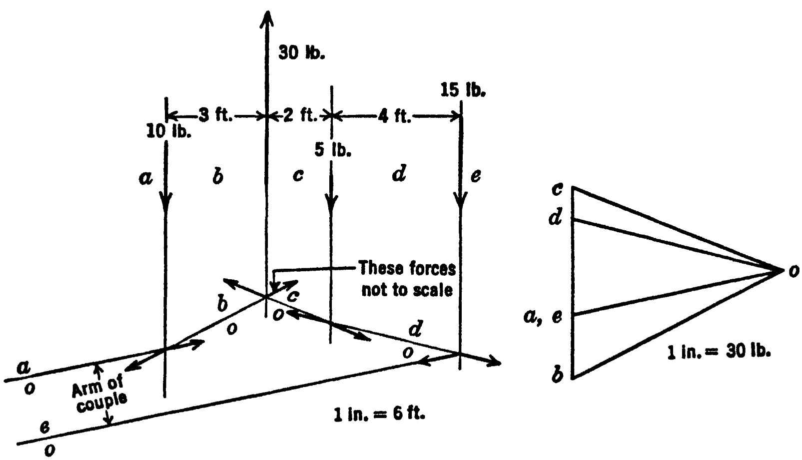


Fig. 6–13

Four parallel forces having the magnitudes and positions shown are to be graphically added. From the force diagram of Fig. 6–13 it will be seen that the resultant force is zero, since the point *e* turns out to be coincident with the point *a*. If we draw the funicular diagram we find that the first and last strings *ao* and *eo*, the intersection of which should give the location of the resultant, do

* Culmann, C., *Die Graphische Statik*, 1864.

not intersect but are parallel, thus indicating that the resultant is a couple. The moment of the couple will be seen to be clockwise, or negative, and the magnitude of the couple will be equal to the magnitude of the component *ao* times the distance between the strings *ao* and *eo*:

$$M = -\left[(1.11 \text{ in.})\left(30 \frac{\text{lb}}{\text{in.}}\right)\right]\left[(0.35 \text{ in.})\left(6 \frac{\text{ft}}{\text{in.}}\right)\right] = -70 \text{ ft-lb}$$

which can for this problem be easily checked analytically.

The foregoing method may also be used for non-parallel force

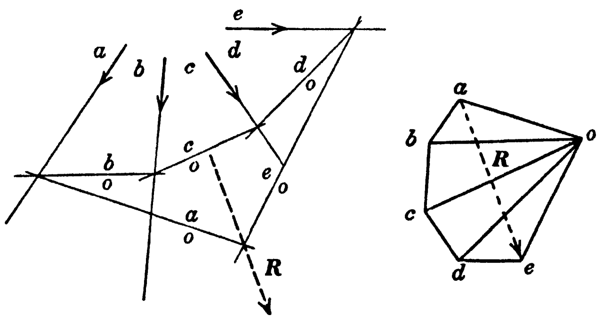
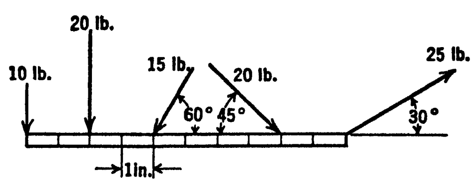


Fig. 6–14

systems as well as for parallel force systems, as may be seen from the example of Fig. 6–14.

PROBLEMS

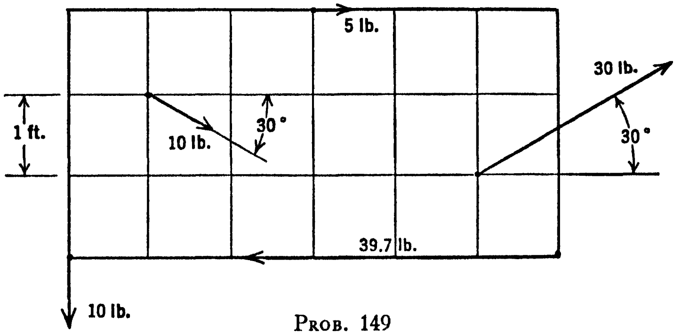
147. Find the resultant of the system shown in Fig. 6–13 by choosing a different position for the pole, and using the graphical method. Check this answer analytically.



Prob. 148

148. A system of forces acts on a beam as shown in the diagram. Find the resultant of the system by the graphical method.

149. Find the resultant of the system of coplanar forces shown in the diagram by the graphical method.



Prob. 149

Equilibrium Problems by Graphical Methods. If a funicular diagram is drawn for a system of forces which are in equilibrium, it will be found that not only does the force diagram form a closed polygon, but that also the two end strings of the funicular diagram will be collinear, forming a closed funicular polygon, as shown in Fig. 6–15.

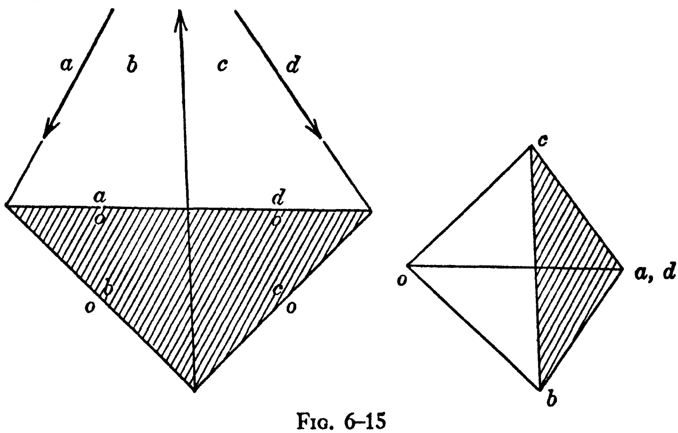


Fig. 6–15

Thus it may be said that the graphical conditions of equilibrium are that both the force polygon and the funicular polygon should close. The closure of the force polygon means that $\Sigma F = 0$ and the closure of the funicular polygon means that $\Sigma M = 0$.

We may use the fact that the two diagrams must close to obtain a graphical solution for equilibrium problems, as can be seen from the following example.

EXAMPLE. Find graphically the reaction forces at *A* and *B* for a truss loaded as shown in Fig. 6–16.

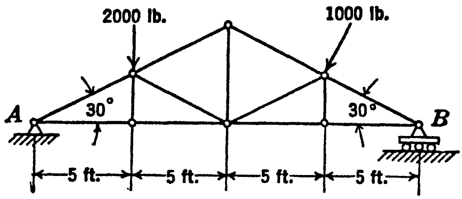


FIG. 6–16

Solution. A free-body diagram of the truss with the forces identified by means of Bow's Notation is shown in Fig. 6–17. The

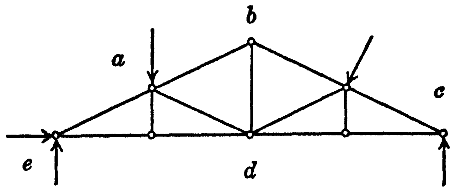


FIG. 6–17

unknown forces *ea*, *cd*, and *de* are shown with their correct lines of action; their magnitudes, of course, cannot as yet be drawn to scale. The notation should be so arranged that the known forces come first, so that the diagrams can be as completely constructed as possible. The diagram is then drawn to scale, and the force polygon and funicular polygon are constructed with the known forces.

In Fig. 6–18 we show as much of the force and funicular polygons as can be drawn with the known forces. The complete solution to scale is shown in Fig. 6–19.

It will be seen from the force polygon of Fig. 6–18 that it is possible to solve for one of the unknown forces, *ea*, directly from the force polygon, since it is known that the two forces *cd* and *de* must lie along a vertical line through *c*, and that the force *ea* must lie along a horizontal line through *a*, for closure of the force polygon. The intersection of these lines fixes the point *e*, which determines the force *ea* and permits us to draw another string, *eo*, in the funicular polygon. The force polygon is now closed, but the funicular polygon is still open, since the string *do*, which would join the points marked *A* and *B* in Fig. 6–19, is missing. For

equilibrium, this string must be present, however, so we draw *do*, which closes the funicular polygon. We can now determine the

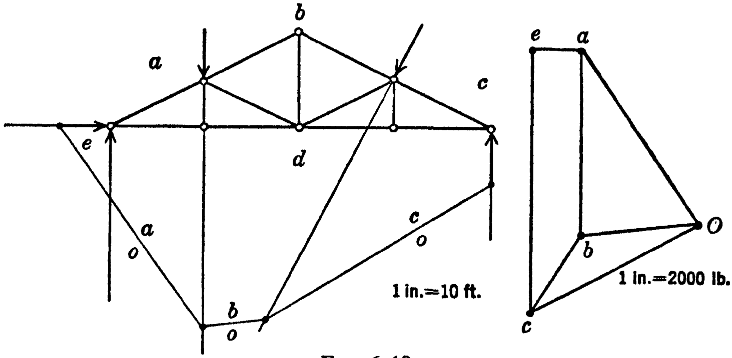


Fig. 6–18

point *d* in the force polygon, because, working backwards, we can draw the ray *do* in the force polygon parallel to the string *do* in the funicular polygon. Having the point *d* in the force polygon

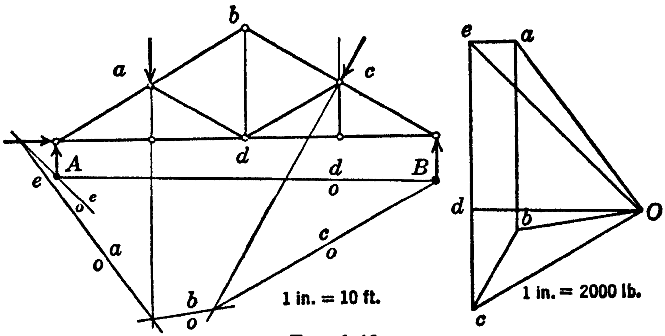


Fig. 6–19

gives us directly the forces *cd* and *de* and hence completes the solution.

Scaling off the values of the forces, we find:

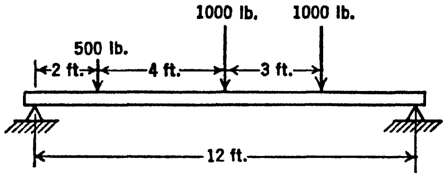
$$ea = 500 \text{ lb}$$
$$cd = 1070 \text{ lb}$$
$$de = 1780 \text{ lb}$$

To check our solution, let us determine the force *cd* analytically by taking moments about the left end of the truss:

$$\Sigma M_A = 0 = (20)(cd) - (5)(2000) - (866)(15) + (500)(2.88)$$
$$cd = 1080 \text{ lb}$$

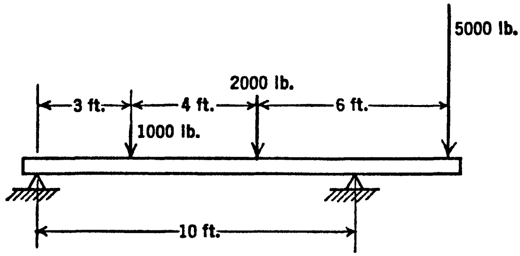
PROBLEMS

150. A beam is supported at two points and is loaded by three parallel forces as shown in the diagram. Find graphically the two reactions at the supports.



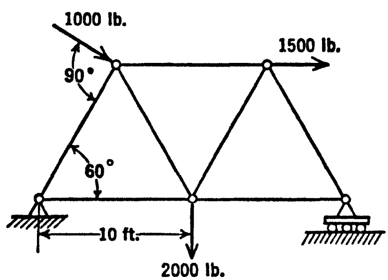
PROB. 150

151. A system of three parallel forces acts on a beam supported as shown in the figure. Find graphically the two reactions at the supports. Check these reactions analytically.



PROB. 151

152. A truss is loaded by three forces as shown in the diagram. Find graphically the forces at the supports. Check the value of the vertical reaction at the right end of the truss analytically.



PROB. 152

The Funicular Polygon as a Moment Diagram. We have seen above that the construction of a funicular polygon is equivalent to solving the moment equations. We shall now show the relationship between the funicular diagram and the moment of a force. In Fig. 6–20a is shown a force (*ab*) as it appears in the funicular diagram. The force diagram for the same force is shown in Fig. 6–20b. In (a) are also shown the two strings *ao* and *bo* of the funicular diagram which are parallel to the rays *ao* and *bo* in the

force diagram. Take any point A at a distance l from the force (ab) and let the intercept of the funicular diagram which would be cut by a line through A parallel to ab be called d. We further call the perpendicular distance between the pole O and the force (ab) the *pole distance*, p. Then, the two triangles which are shown shaded in Fig. 6–20 will be seen to be similar, and we have:

$$\frac{l}{p} = \frac{d}{(ab)}$$

$$l(ab) = pd$$

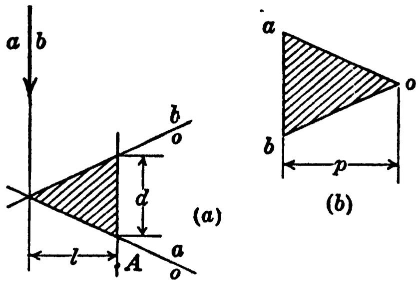


FIG. 6–20

but $l(ab)$ = the moment of the force (ab) about the point A; therefore, we can say:

The moment of a force about a point is equal to the pole distance times the intercept on the funicular diagram.

The chief value of this proposition may be seen in the case of a system of parallel forces, as shown in Fig. 6–21.

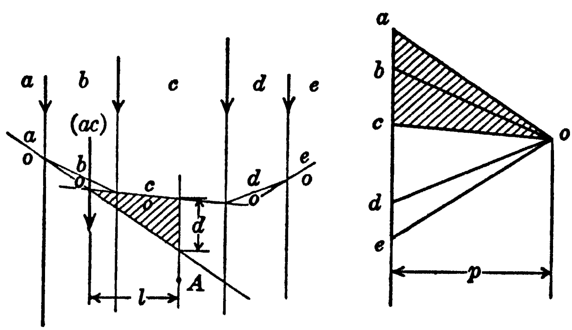


FIG. 6–21

Suppose that we wish to find the sum of the moments of all of the forces to the left of the point A about the point A. As before, we let d be the intercept on the funicular diagram of a line through A parallel to the forces of the system. The sum of the moments of the forces (ab) and (bc) about the point A will be equal to the moment of the resultant of the two forces (ab) and (bc) about the point A. This resultant is the force (ac), which acts at the position shown on the funicular diagram. Again noting that the

shaded triangles shown in the funicular diagram and in the force diagram are similar, we have:

$$\frac{l}{p} = \frac{d}{(ac)}$$
$$l(ac) = pd$$

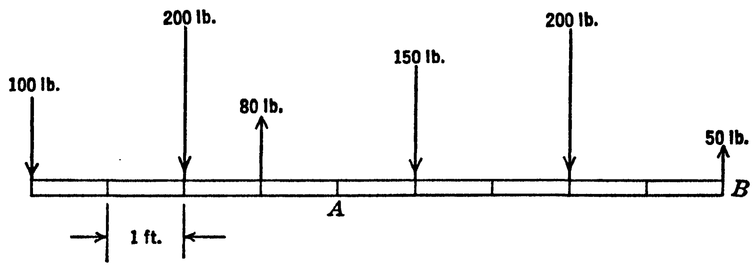
or the sum of the moments to the left of the point A is equal to the intercept in the funicular diagram times the pole distance.

It will be seen from Fig. 6–21 that this conclusion would be true for any position of the point A and for any number of forces.

In a number of engineering problems it is required that a diagram be prepared which will show, for a system of parallel forces, the sum of the moments of all of the forces to the left of a point for various positions of the point. It will be seen that the funicular diagram represents, to a scale depending on the pole distance p, such a moment diagram. Funicular diagrams are much used, in this way, in the graphical determination of the deflections and critical speeds of shafts.

PROBLEMS

153. Find graphically the resultant moment about the point A and about the point B of the forces to the left of these points, for the

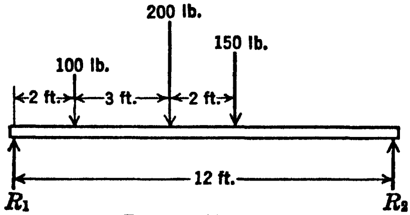


PROB. 153

system shown in the diagram. Check the moments about A analytically.

154. A beam loaded as shown in the diagram is supported at its two ends. Find graphically by means of a funicular diagram the point along

the length of the beam about which the sum of the moments of all of the forces on one side of the point is a maximum.



PROB. 154

The Graphical Analysis of a Truss. We have considered above two analytical methods of determining the forces acting in the members of a truss, the method of joints and the method of sections. In solving the equations which were written for the various concurrent force systems encountered in the method of joints, we used analytical methods. It would also be possible to use a graphical method, i.e., a graphically constructed force polygon, to find the forces involved in those concurrent force systems. The following example will show how this graphical method will apply to the solution of a typical truss problem. Suppose that we consider the same truss used above as an example of the analytical method and shown in Fig. 6–22. We further suppose that we have already solved for the reactions at the supports, either analytically, as in the above example, or graphically by means of

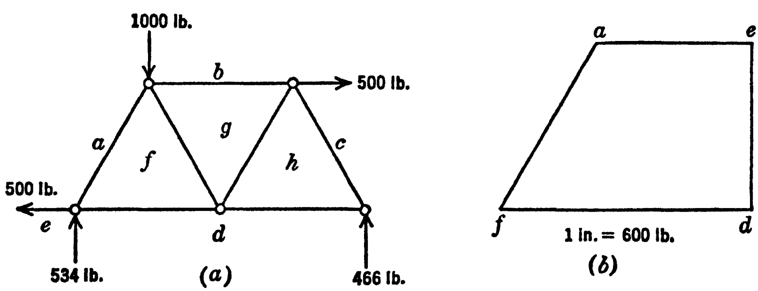


FIG. 6–22

a funicular diagram. We have thus the situation shown in Fig. 6–22a.

To identify the various forces, we use Bow's notation, and letter the spaces between the forces in the space diagram, as in

Fig. 6–22a. The force polygon for the left-hand joint is shown in Fig. 6–22b and will illustrate the use of Bow's notation for this type of diagram. We adopt the convention of identifying the forces acting at any particular joint by reading the letters of Bow's notation clockwise around the joint. By maintaining this consistency of direction, the signs of the forces can be automatically determined. For example, if we are referring to the left-hand joint, the force in the inclined member is (*af*), whereas for the upper left-hand joint the force in this same member is (*fa*). Since these two forces should actually be equal and opposite, it will be seen that the above convention will lead to a consistent picture of the signs of the various forces. By reading the forces in the force polygon in their proper order, referred to any joint, it can be determined whether a particular member is in tension or compression. For example, in Fig. 6–22b, reading the letters clockwise, we see that the force (*af*) is directed toward the left-hand joint and is hence a compressive force, whereas the force (*fd*) is directed away from the joint and is hence a tension force.

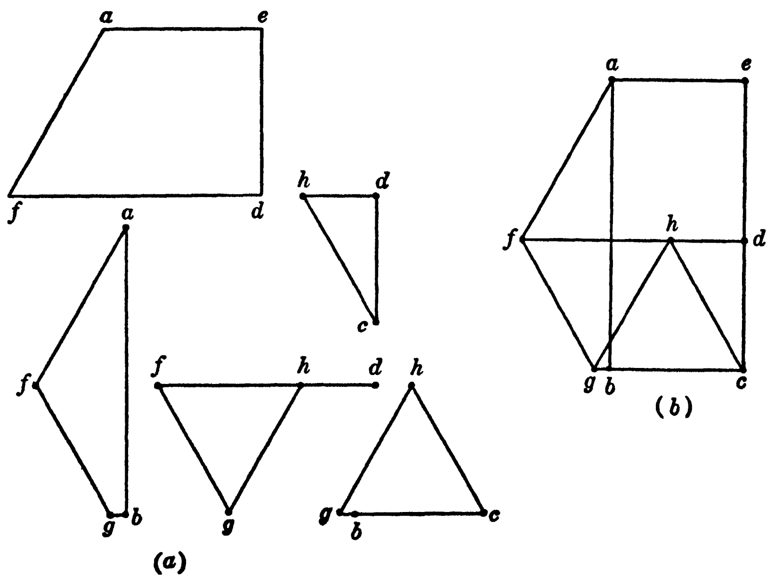


Fig. 6–23

In Fig. 6–23a are shown the complete force diagrams for each of the joints of the truss shown in Fig. 6–22a. By examining these

figures, it will be seen that many of the lines have been repeated from diagram to diagram, so that the construction could be considerably simplified if the diagrams were combined into one diagram for the whole truss, as shown in Fig. 6–23b. The combined force diagram obtained in this way is called the *Maxwell-Cremona** diagram and such diagrams are frequently used for the analysis of trusses. It will be evident that such a complete diagram could be drawn at once for any truss which the method of joints will solve. As a second example of the appearance of these Maxwell-Cremona diagrams, Fig. 6–24 may be consulted.

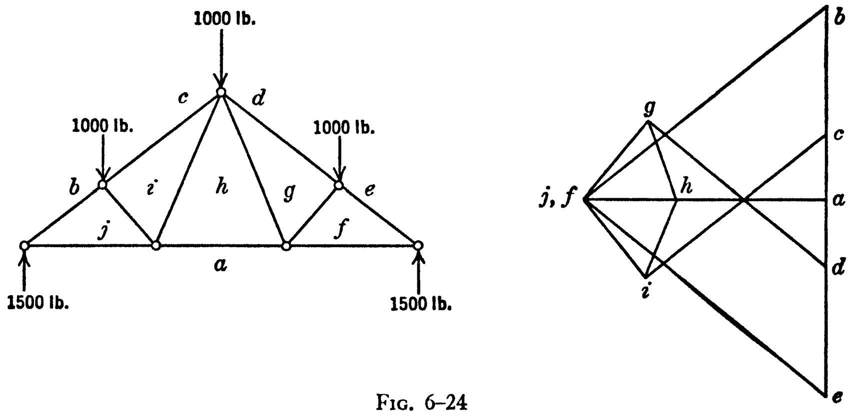


FIG. 6–24

It should be noted that it is not necessary when constructing the Maxwell-Cremona diagram to draw the force polygon for each joint of the truss. The points of the diagram can all be located by starting from the known points and drawing lines parallel to the truss members through these known points. The intersection of these lines will determine the unknown points.

PROBLEMS

155. Find graphically the forces in the members of the truss shown in Problem 141 by drawing the Maxwell-Cremona diagram. Compute analytically the reactions needed to start the diagram.

156. Same for Problem 142.

157. Same for Problem 145.

158. Same for Problem 146.

* Maxwell, J. C. "On Reciprocal Figures and Diagrams of Forces," *Phil. Mag.*, Vol. XXVI, 1864, p. 250.

Beams and Beam Reaction Forces. The structures hitherto considered in the present chapter have been of the type in which the members have been subjected to axial loads of tension or compression. Another very common type of structural member is a *beam*, which is defined as a member whose loads are primarily transverse to its axis rather than in the direction of the axis. These transverse loads cause a bending of the beam, instead of the extensions and compressions associated with the axial loads. The forces which are exerted on the beam by its supports are called the *beam reactions*, to distinguish them from the active forces which form the useful load on the beam. The computation of these reaction forces is the first step in the design of any beam.

The type of reactive force system set up will depend upon the way in which the beam is supported. We shall limit our present discussion to statically determinate supports, so that the reaction forces will in all cases be determined completely from the equations of statics applied to the whole beam as a free body.

The two most common types of beams are the *simply supported* beam on two supports and the *cantilever* beam, as shown in Fig. 6–25.

Considering only loads normal to the beam axis, the simply supported beam of Fig. 6–25a is supported in such a way that the

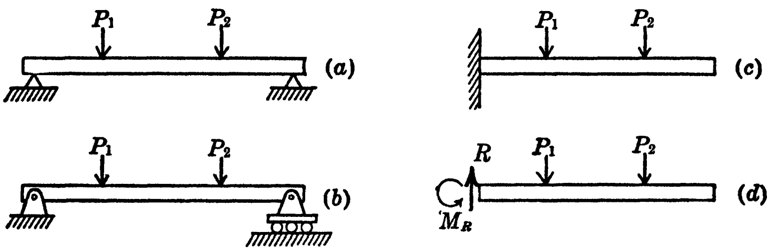


Fig. 6–25

reactive forces are normal to the beam. If one end of the beam is pinned, for example, the other end must be supported on frictionless rollers, as in Fig. 6–25b, so that no reactive force parallel to the beam can be developed. The cantilever beam of Fig. 6–25c is built in at one end in such a way that the direction of the tangent to the beam at that point is fixed. This requires that the support exert a moment on the beam, as is indicated in Fig. 6–25d.

Shearing Forces and Bending Moments in Beams. In order to determine the ability of the beam to withstand safely the applied loads, it is necessary to know the distribution of the internal forces in the beam. The complete solution of this problem is given in books on the Theory of Elasticity and Strength of Materials, but the necessary first steps in the solution involve only the principles of statics.

To find the internal forces in a beam at a particular section, one can imagine that the beam is cut at that section, and a free-body diagram of one portion of the beam can be drawn. Consider, for example, the simply supported beam with a single concentrated

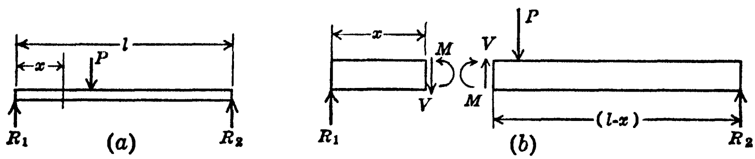


FIG. 6–26

load, as shown in Fig. 6–26a, and suppose that we wish to know the internal forces in the beam at a section a distance x from the left reaction. In Fig. 6–26b the beam is shown cut at the section x into two free-body diagrams. The actual forces acting on the cut cross section can be represented as in Example 4, p. 34, but for our present purpose all that we require is the resultant of all of the forces normal to the beam axis, which we shall call the *shearing force* V in the beam, and the moment of the couple formed by the compression forces on the top portion of the beam and the tension forces on the bottom portion of the beam, which we shall call the *bending moment* M in the beam. Our present purpose is the determination of this shearing force and this bending moment for each section of the beam. From the complete free-body diagram of the beam shown in Fig. 6–26a, the two reaction forces R_1 and R_2 can be found. Thus, the free-body diagram of either end of the beam in Fig. 6–26b will contain only the two unknowns V and M, which can therefore be found directly from the equations of statics. To decide on a sign convention for V and M, we note that the shearing force on the left section is directed downward, while the shearing force on the right section, being the equal and opposite reaction of the other force, is directed upward. In order,

however, to have one sign for the shearing force at one point in the beam, the convention is adopted that, if the shear in a beam is in such a direction that the right portion of the beam would tend to move downward with respect to the left portion, the shearing force at that point is positive. For a similar reason, a bending moment which tends to bend the beam concave upward is called a *positive* bending moment. These conventions are illustrated in Fig. 6–27.

The way in which the shearing force and the bending moment vary along a beam is usually shown by drawing a diagram beneath the beam as in the following examples. It should be noted that the same shearing force and bending moments will be computed for both portions of the beam. One usually selects as a free-body diagram the portion of the beam which involves the smaller number of forces.

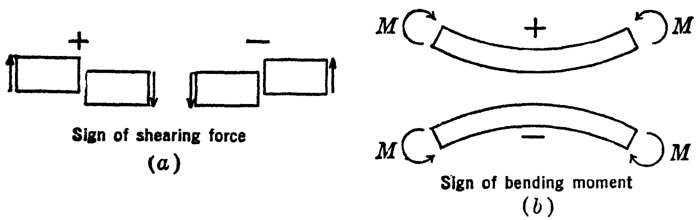


FIG. 6–27

The bending moment diagram can be drawn directly by a graphical method from the known beam loadings. This method follows directly from the discussion of the graphical determination of a bending moment by means of a funicular polygon, as given on p. 160. Since by definition the bending moment at a point in a beam is the sum of all of the moments of all of the forces on the beam to one side of the point, it follows that the funicular polygon as drawn in Problem 154, is the bending moment diagram for the beam.

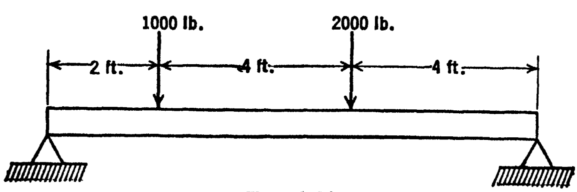


FIG. 6–28

EXAMPLE (1) Draw the shearing force and bending moment diagrams for the simply supported beam with two concentrated loads shown in Fig. 6–28.

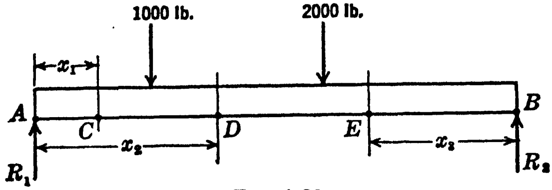


FIG. 6–29

Solution. From a free-body diagram (Fig. 6–29) of the whole beam, the reaction forces can be found:

$$\Sigma M_A = 0 = -(2)(1000) - (6)(2000) + (10)(R_2); \quad R_2 = 1400 \text{ lb}$$
$$\Sigma F_y = 0 = R_1 + 1400 - 1000 - 2000; \quad R_1 = 1600 \text{ lb}$$

To find the shearing force at any point between R_1 and the 1000-lb load, a free-body diagram (Fig. 6–30) of the portion of the beam to the left of section C is drawn:

$$\Sigma F_y = 0 = 1600 - V_x; \quad V_x = 1600 \text{ lb}$$
$$\Sigma M_C = 0 = M_x - 1600x_1;$$
$$M_x = 1600x_1 \text{ ft-lb}$$

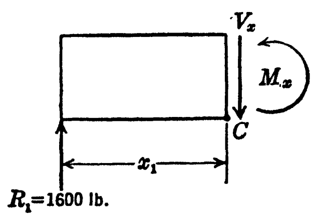


FIG. 6–30

For the next section of the beam, between the 1000-lb load and the 2000-lb load, we draw a free-body diagram (Fig. 6–31) of the portion of the beam to the left of section D.

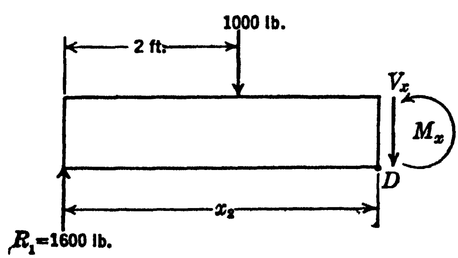


FIG. 6–31

$$\Sigma F_y = 0 = 1600 - 1000 - V_x; \quad V_x = 600 \text{ lb}$$
$$\Sigma M_D = 0 = M_x - 1600x_2 + 1000(x_2 - 2)$$
$$M_x = 600x_2 + 2000 \text{ ft-lb}$$

For the rest of the beam, we draw a free-body diagram (Fig. 6–32) of that portion of the beam to the right of section E:

$$\Sigma F_y = 0 = 1400 + V_x; \quad V_x = -1400 \text{ lb}$$
$$\Sigma M_E = 0 = 1400x_3 - M_x; \quad M_x = 1400x_3 \text{ ft-lb}$$

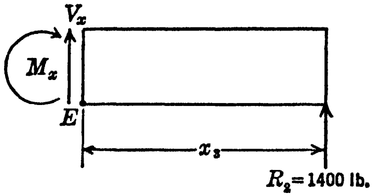


FIG. 6–32

Note that in all of the free-body diagrams the unknown shearing force and bending moments have been shown in the positive direction. In writing the equations of statics, the usual sign conventions have been used.

The shearing force and the bending moment diagrams can now be drawn for the whole beam as follows (Fig. 6–33):

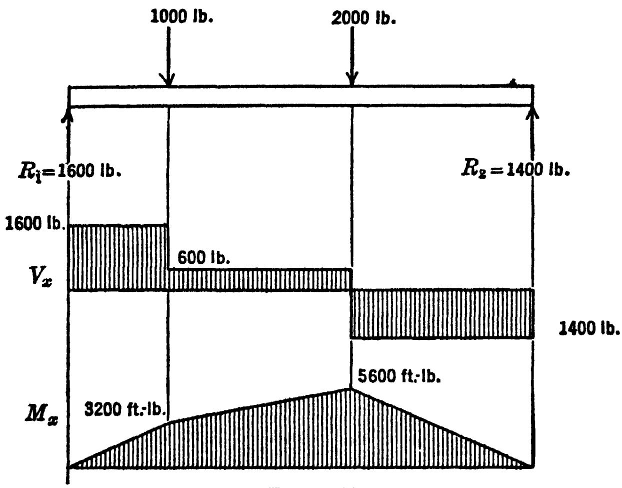


FIG. 6–33

EXAMPLE (2) Draw the shearing force and bending moment diagrams for the cantilever beam with the uniformly distributed load shown in Fig. 6–34.

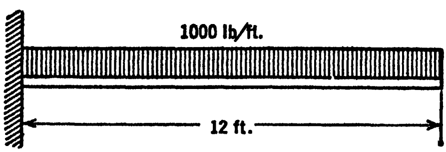


FIG. 6–34

Solution. From the free-body diagram of the entire beam, the reaction force and the reaction moment can be determined (Fig. 6–35):

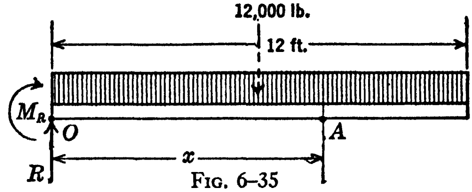


FIG. 6–35

$$\Sigma M_0 = 0 = -(12{,}000)(6) - M_R$$
$$M_R = -72{,}000 \text{ ft-lb}$$
$$\Sigma F_y = 0 = R - 12{,}000; \quad R = 12{,}000 \text{ lb}$$

To find the shearing force and the bending moment at any point in the beam, we draw a free-body diagram of the portion of the beam to the left of the section A, located a distance x from the left end of the beam (Fig. 6–36):

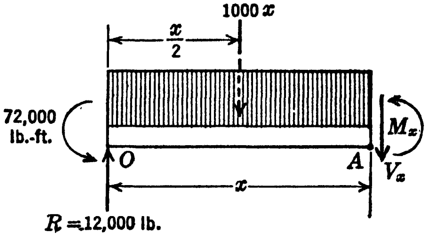


FIG. 6–36

$$\Sigma F_y = 0 = 12{,}000 - 1000x - V_x$$
$$V_x = 12{,}000 - 1000x \text{ lb}$$
$$\Sigma M_A = 0 = M_x + (1000x)\left(\frac{x}{2}\right) + 72{,}000 - 12{,}000x$$
$$M_x = 12{,}000x - 500x^2 - 72{,}000 \text{ ft-lb}$$

Plotting these expressions gives (Fig. 6–37):

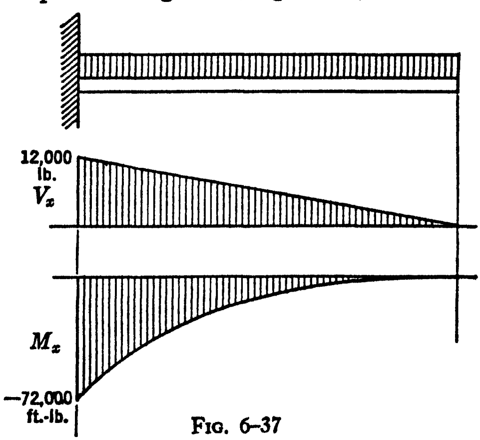


FIG. 6–37

For this problem, it would have been better to have measured x from the right end of the beam, and to have drawn a free-body diagram of that portion of the section (Fig. 6–38). This would have eliminated the necessity of determining the beam reactions:

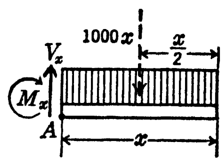


FIG. 6–38

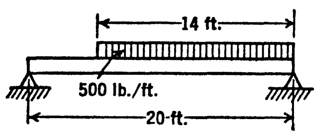
$$\Sigma F_y = 0 = V_x - 1000x; \quad V_x = 1000x$$
$$\Sigma M_A = 0 = -M_x - 500x^2; \quad M_x = -500x^2$$

It will be seen that these expressions give the same shearing force and bending moment diagrams as found above.

PROBLEMS

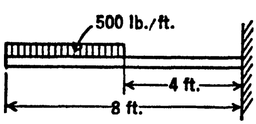
For the following beams, draw shearing force and bending moment diagrams, giving numerical values at various significant points:

(159)



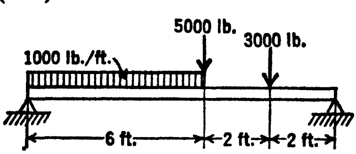
PROB. 159

(160)



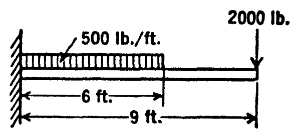
PROB. 160

(161)



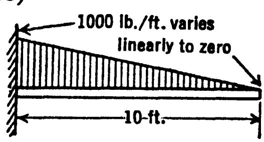
PROB. 161

(162)



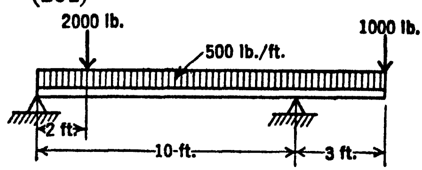
PROB. 162

(163)



PROB. 163

(164)



PROB. 164

Relationship between Shear Force and Bending Moment Diagrams. A general relationship between the load, the shearing force, and the bending moment may be derived in the following way. Consider a beam with any load which is given by the func-

tion $q(x)$, where q is a distributed load in pounds per foot which may vary in any way. In Fig. 6–39 is shown a free-body diagram of a portion of this beam of length dx.

For equilibrium, we have:

$$\Sigma F_y = 0 = V + q\,dx - (V + dV)$$

$$\Sigma M_0 = 0 = M + dM - M - V\,dx - (q\,dx)\left(\frac{dx}{2}\right)$$

From the first equation:

$$q = \frac{dV}{dx}$$

And from the second equation, neglecting the second-order differential $(dx)(dx)$:

$$V = \frac{dM}{dx}$$

These expressions give the relationship between the load, shearing force, and bending moment in a beam. The first expression states that the ordinate of the load diagram is equal to the slope of the shearing force diagram, while the second expression states that the ordinate of the shear force diagram is equal to the slope of the bending moment diagram. By the use of the second statement, the bending moment diagram can be constructed once the shearing force diagram is known. The value of the bending moment can be expressed in terms of the shearing force by integrating the second expression above to give:

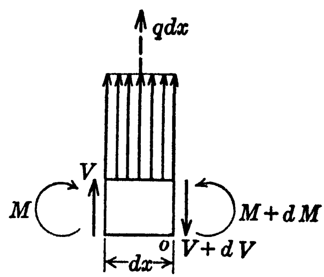


FIG. 6–39

$$M = \int V\,dx + c$$

This states that the difference between the magnitude of the bending moment at two points of a beam is equal to the area of the shearing force diagram between those two points.

In the design of beams, one is usually most interested in finding the magnitude of the maximum bending moment. This maximum bending moment will occur at the point where the slope of the bending moment diagram is zero or where there is a discontinuity

in the slope. Each point in a beam where the shearing force diagram passes through zero or has a multiple value should thus be examined as a possible point of maximum bending moment.

EXAMPLE. Using the information derived above, find the maximum bending moment in the beam shown in Fig. 6–40.

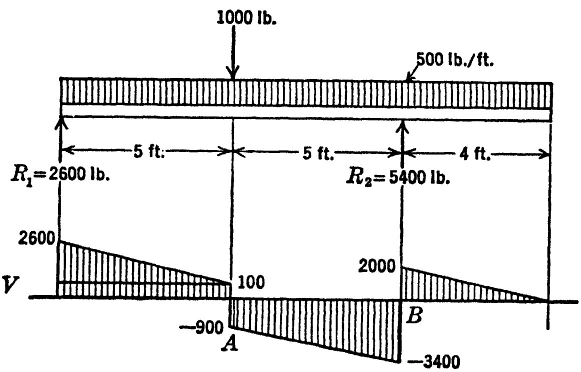


FIG. 6–40

Solution. We first determine the reactions and plot the shearing force diagram by the previously described method. The relationship $q = \frac{dV}{dx}$ tells us that where the load is uniformly distributed the slope of the shear force diagram is constant, and that where a concentrated load occurs there will be a discontinuity in the shear force diagram.

The shear force diagram crosses the zero axis at two points, A and B, one of which must be the maximum bending moment for the entire beam. The magnitude of the bending moment at A is the area of the shear force diagram to the left of A, and the magnitude of the bending moment at B is the area of the shear force diagram to the right of B. These areas are:

$$\text{at } A: \quad (2500)(5)(\tfrac{1}{2}) + (5)(100) = 6750 \text{ ft-lb}$$
$$\text{at } B: \quad -(2000)(4)(\tfrac{1}{2}) = -4000 \text{ ft-lb}$$

Hence the maximum bending moment is the positive bending at A of 6750 ft-lb.

PROBLEMS

165. Find the bending moment diagram for Example 1, p. 169, graphically, using a funicular polygon, and check the maximum value of bending moment in the beam.

166. Using the method of the Example on p. 174, construct the bending moment diagram and find the maximum bending moment for the beam shown in Problem 159.

167. Same as Problem 166 above for the beam of Problem 161.

168. Same as Problem 166 above for the beam of Problem 164.

Flexible Cables. A commonly encountered structural element is a flexible cable or chain, suspended from its two ends and loaded by a system of vertical forces. By a flexible cable we mean a cable that is capable of supporting only a tension force which at any point is tangent to the curve assumed by the cable. The two most common types of loading are a load which is uniformly distributed across the horizontal span of the cable, and a vertical load which is uniformly distributed along the length of the cable. The first case is encountered in suspension bridges where the weight of the horizontally supported structure may be large compared to the weight of the suspension cables, whereas the second type of loading is that encountered by electric transmission lines.

We shall first see how far we can carry the analysis for a flexible cable whose ends are supported at different heights, and which is loaded by a system of vertical forces that vary in any given

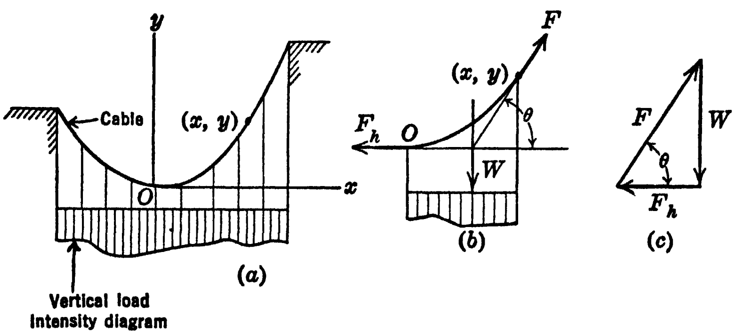


Fig. 6–41

manner, as shown in Fig. 6–41a. We shall neglect the stretching of the cable under the load by assuming that the cable is inextensible.

We fix the origin of an xy coordinate system, which is in the plane of the cable, at the lowest point of the cable. We then draw a free-body diagram of the portion of the cable between this lowest point O, at which the cable has a horizontal tangent, and any point x, y on the cable. The forces acting on the cable are the horizontal force F_h at O, the tangential force F at the point x, y and the force W, which is the portion of the load acting upon the length of cable from O to x, y and which includes the weight of the cable. This force W acts through the center of gravity of the load intensity diagram, as shown in Fig. 6–41b.

We now note that we have a system of three forces in equilibrium, so that they must be concurrent. Hence, if the shape of the load intensity diagram is known, the position of W will be known, and the direction of the tangential force F will be fixed. Since the direction of the tangent to the cable at any point could thus be determined, the shape of the curve assumed by the cable could be found.

If we let θ be the angle made by the tangent to the curve at the point x, y with the horizontal, we have:

$$\frac{dy}{dx} = \tan \theta$$

Also from the force diagram of Fig. 6–41c we note that:

$$\tan \theta = \frac{W}{F_h}$$

so that the differential equation describing the shape of the suspended cable is:

$$\frac{dy}{dx} = \frac{W}{F_h}$$

For the two simple load distributions mentioned above, this differential equation can be solved analytically, and the equation of the suspended cable can be found.

Horizontally Distributed Load. Suppose that a uniformly distributed load of intensity w lb per ft acts upon the cable. Then referring to the diagram of Fig. 6–41b, it will be seen that $W = wx$, and the differential equation becomes:

$$\frac{dy}{dx} = \frac{wx}{F_h}$$

This equation can now be integrated to give:

$$y = \int \frac{wx}{F_h}\,dx + c = \frac{wx^2}{2F_h} + c$$

To find the constant of integration we note that for the coordinate system chosen $y = 0$ when $x = 0$ so that $c = 0$, then

$$y = \frac{wx^2}{2F_h} \tag{1}$$

This is the equation of a parabola, so that we have shown that a flexible cable loaded by a uniformly distributed vertical load will assume a parabolic form.

From the force diagram of Fig. 6–41c we have:

$$F = \sqrt{F_h^2 + W^2}$$

so that for the uniform horizontal load

$$F = \sqrt{F_h^2 + w^2x^2} \tag{2}$$

From this it appears that the horizontal force F_h at the lowest point in the cable is the smallest force in the cable, and that this force increases as x increases, becoming a maximum at the support.

In most practical problems we will know the span of the cable, i.e., the horizontal distance between supports, and we will wish to know the relationship between the maximum force in the cable and the vertical sag of the cable. We shall work out this relationship for the case in which the two ends of the cable are supported at the same level as in Fig. 6–42. Using the same coordi-

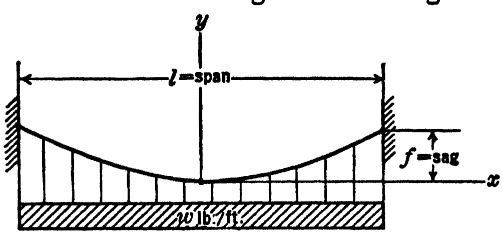


Fig. 6–42

nate system used in the development of Equation (1) above, we note that when $x = \frac{l}{2}$, $y = f$, so that we have from Equation (1):

$$f = \frac{w\left(\frac{l}{2}\right)^2}{2F_h}; \quad F_h = \frac{wl^2}{8f}$$

Then the maximum force in the cable, at the support, is found from Equation (2).

$$F_{max} = \sqrt{\left(\frac{wl^2}{8f}\right)^2 + w^2\left(\frac{l}{2}\right)^2}$$
$$= \frac{wl}{2}\sqrt{1 + \frac{l^2}{16f^2}}$$

Another relationship often required is that between the span l, the total length S of the cable, and the sag f. In the above cable the length would be:

$$S = 2\int_0^{\frac{l}{2}} dS = 2\int_0^{\frac{l}{2}} \sqrt{dx^2 + dy^2} = 2\int_0^{\frac{l}{2}} \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx$$
$$S = 2\int_0^{\frac{l}{2}} \sqrt{1 + \frac{w^2x^2}{F_h^2}}\, dx$$

From the integral tables, we have:

$$S = \frac{F_h}{w}\left[\frac{wx}{F_h}\sqrt{1 + \frac{w^2x^2}{F_h^2}} + \sinh^{-1}\frac{wx}{F_h}\right]\Bigg|_0^{\frac{l}{2}}$$
$$= \frac{l}{2}\sqrt{1 + \left(\frac{wl}{2F_h}\right)^2} + \frac{F_h}{w}\sinh^{-1}\left(\frac{wl}{2F_h}\right)$$

In order to put this into a more convenient form for computation, let us expand each term in a series. The series expansions for these functions are:

$$\sqrt{1 + \left(\frac{wl}{2F_h}\right)^2} = 1 + \frac{1}{2}\left(\frac{wl}{2F_h}\right)^2 - \frac{1}{8}\left(\frac{wl}{2F_h}\right)^4 + \cdots$$
$$\sinh^{-1}\left(\frac{wl}{2F_h}\right) = \left(\frac{wl}{2F_h}\right) - \frac{1}{6}\left(\frac{wl}{2F_h}\right)^3 + \frac{3}{40}\left(\frac{wl}{2F_h}\right)^5 + \cdots$$

Expressing these in terms of the "sag ratio" $\left(\frac{f}{l}\right)$ we have:

$$F_h = \frac{wl^2}{8f}, \quad \text{so that} \quad \left(\frac{wl}{2F_h}\right) = 4\left(\frac{f}{l}\right)$$

This ratio will ordinarily be sufficiently small so that the above series expansions will converge rapidly, and only the first few terms of the series need be retained. Thus:

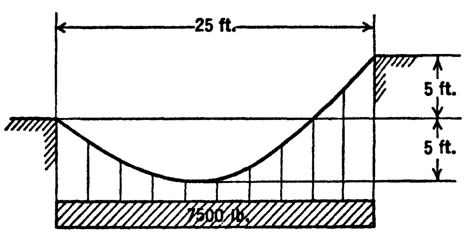
$$S = \frac{l}{2} + \frac{l}{4}\left(4\frac{f}{l}\right)^2 - \frac{l}{16}\left(4\frac{f}{l}\right)^4 + \cdots$$
$$+ \frac{l}{2}\frac{1}{\left(4\frac{f}{l}\right)}\left[\left(4\frac{f}{l}\right) - \frac{1}{6}\left(4\frac{f}{l}\right)^3 + \frac{3}{40}\left(4\frac{f}{l}\right)^5 + \cdots\right]$$
$$= \left(\frac{l}{2} + \frac{l}{2}\right) + \left(4l - \frac{16}{12}l\right)\left(\frac{f}{l}\right)^2 + l\left(-\frac{256}{16} + \frac{768}{80}\right)\left(\frac{f}{l}\right)^4 + \cdots$$
$$= l\left[1 + \frac{8}{3}\left(\frac{f}{l}\right)^2 - \frac{32}{5}\left(\frac{f}{l}\right)^4 + \cdots\right]$$

From this expression the length of the cable can be determined when the span and sag are known.

PROBLEMS

169. The force in a cable which is suspended from two points 50 ft apart on the same level is to be limited to 10,000 lb. What is the maximum total load which the cable can support, if the load is uniformly distributed horizontally, and if the sag is 10 ft?

170. A cable 100 ft long is suspended from two points which are on the same level and which are 95 ft apart. If the cable supports a total load of 10,000 lb, uniformly distributed along the horizontal, what is the maximum force in the cable?



PROB. 171

171. A uniformly distributed horizontal load of 7500 lb is supported by a cable as shown in the diagram. Find the maximum force in the cable.

Cable Loaded by Its Own Weight. In some cases the load acting on a flexible cable is uniformly distributed along the length of the cable rather than along the horizontal. This is always the situation when the major load on the cable is its own weight, as

for an electrical transmission line. In this problem the same differential equation derived above is valid:

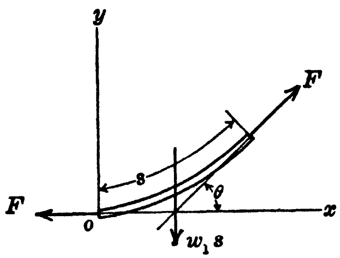


FIG. 6–43

$$\frac{dy}{dx} = \frac{W}{F_h}$$

where now

$$W = w_1 s$$

where w_1 is the weight per unit length of the cable, and s is the length of the cable from the lowest point to the point x, y.

Expressing s in terms of x and y we have:

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

Since

$$\frac{dy}{dx} = \frac{w_1 s}{F_h}$$

this becomes

$$\frac{ds}{dx} = \sqrt{1 + \left(\frac{w_1 s}{F_h}\right)^2}$$

$$x = \int \frac{ds}{\sqrt{1 + \left(\frac{w_1 s}{F_h}\right)^2}}$$

From the integral tables we find:

$$x = \frac{F_h}{w_1} \sinh^{-1} \frac{w_1 s}{F_h} + C_1$$

when $x = 0$, $s = 0$, $\therefore C_1 = 0$. Thus

$$\frac{w_1 s}{F_h} = \sinh \frac{w_1 x}{F_h}; \quad s = \frac{F_h}{w_1} \sinh \frac{w_1 x}{F_h}$$

Substituting this into the original differential equation, we get:

$$\frac{dy}{dx} = \sinh \frac{w_1 x}{F_h}$$

$$y = \int \sinh \frac{w_1 x}{F_h}\, dx = \frac{F_h}{w_1} \cosh \frac{w_1 x}{F_h} + C_2$$

when $x = 0$, $y = 0$

$$0 = \frac{F_h}{w_1} \cosh 0 + C_2; \quad C_2 = -\frac{F_h}{w_1}$$

hence

$$y = \frac{F_h}{w_1}\left(\cosh \frac{w_1 x}{F_h} - 1\right)$$

This equation gives the curve which will be assumed by the flexible cable hanging under its own weight. Such a curve is called a *catenary*.

The force in the cable can be found in the same way as for the parabolic cable, Equation (2) above becoming:

$$F = \sqrt{F_h^2 + w_1^2 s^2} = F_h \sqrt{1 + \sinh^2 \frac{w_1 x}{F_h}}$$

since

$$\cosh^2 \alpha - \sinh^2 \alpha = 1$$

this gives:

$$F = F_h \cosh \frac{w_1 x}{F_h}$$

for the case of a cable suspended from two supports at the same level. The maximum force will occur at the support, where $x = \frac{l}{2}$, so

$$F_{max} = F_h \cosh \frac{w_1 l}{2F_h}$$

To find F_h we note that when $x = \frac{l}{2}$, $y = f$. Introducing these values into the equation relating x and y gives:

$$f = \frac{F_h}{w_1}\left(\cosh \frac{w_1 l}{2F_h} - 1\right)$$

This equation can in most practical problems be solved easily by trial and error.

PROBLEMS

172. A cable 1000 ft long which weighs 2 lb per ft is to be suspended from two points at the same level. If the force in the cable is to be limited to 2000 lb, what should be the span?

173. Compute the sag for the cable of Problem 172 above. Find the maximum force in this cable, for the computed span and sag, by assuming that the total weight of the cable is uniformly distributed horizontally, and compare this with the actual force.

174. A cable weighing 3 lb per ft is stretched between two points on the same level 1000 ft apart. The sag is 100 ft. Find the maximum tension in the cable, and the length of the cable. (Note: This problem requires a trial and error solution.)

175. A transmission line is to be strung with a minimum clearance of 30 ft to the ground. The wire used weighs $\frac{1}{2}$ lb per ft, and the maximum permissible tension force is 1000 lb. If the towers are 40 ft high, how far apart should they be?

176. A flexible cable 300 ft long weighs 1 lb per ft. The cable is suspended from two supports at the same level that are 200 ft apart. Find the sag of the cable and the maximum force in the cable.

177. In the formula derived above for the relationship between x and y for the catenary cable, show that if the first two terms only of the series expansion for the hyperbolic cosine are retained, the expression becomes identical to that found for the parabolic cable.

CHAPTER VII

INTRODUCTION TO STATICALLY INDETERMINATE STRUCTURES

. . . we may express the elasticity of any substance by the weight of a certain column of the same substance, which may be denominated the modulus of its elasticity, and of which the weight is such that any addition to it would increase it in the same proportion as the weight added would shorten, by its pressure, a portion of the substance of equal diameter.—Thomas Young, "A Course of Lectures on Natural Philosophy and the Mechanical Arts" (1807).

As we have seen in a preceding chapter, there are some equilibrium problems for which the principles of statics alone are not sufficient to give a solution. If a structure, for example, has more constraints than are required for statically determinate constraint of the system, there will not be sufficient statics equations to determine the constraining forces. Such problems are called statically indeterminate problems, and in the present chapter we shall show, in the simplest case, how such problems can be solved.

In all of the problems hitherto discussed in this book, it has not been necessary to consider the deformations of the bodies caused by the applied forces. For statically indeterminate problems, however, it is a knowledge of the deformations of the system which supplies the additional relationships required for the solution.

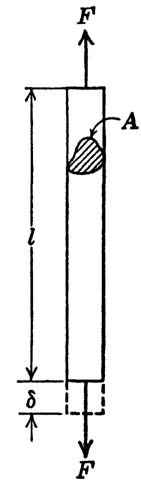


FIG. 7–1

Stress and Strain in Tension and Compression. Consider a straight bar of length l and of uniform cross-sectional area A, as shown in Fig. 7–1. Experiment shows that if a tension force F is applied along the axis of the bar, the bar will be extended by an amount δ, and that if the force is removed, the bar will return to its original length. It is further found that the same force F applied as a compressive force will cause a shortening of the bar

by the same amount. A material which exhibits this property of returning to its original dimensions after loads have been removed is said to be perfectly *elastic*. Experiment also shows that the total deformation δ is directly proportional to the length of the bar, inversely proportional to the cross-sectional area of the bar, and that it also depends upon the material of which the bar is made. These empirical relationships are summed up in the following expression:

$$\delta = \frac{Fl}{AE}$$

where E is the significant property of the material, called the *modulus of elasticity*, or *Young's Modulus*. The modulus of elasticity varies widely for different materials as may be seen from the following table:

Modulus of Elasticity

Material	E (*lb per sq in.*)
Steel	30×10^6
Aluminum	10×10^6
Copper	17×10^6
Brass	14×10^6
Tungsten	60×10^6

The above expression can be written in a somewhat more general way by introducing two new terms; the unit deformation $\frac{\delta}{l} = \epsilon$, called the *strain*, and the unit load $\frac{F}{A} = \sigma$, called the *stress*. With this notation we have

$$\sigma = E\epsilon$$

This linear relationship between stress and strain is called *Hooke's Law*. It should be noted that Hooke's Law holds only within certain limits of stress, and that even within those limits the law is only approximate for most materials. The elastic properties of materials are usually given in the form of a stress-strain curve, as shown in Fig. 7–2. It will be noted in this figure that the properties of various common materials are considerably different, and that while some materials such as steel show a definite region for which Hooke's Law holds, others, such as aluminum, have no strictly linear relationship. For all metals, however, the

assumption is made that for low stresses the stress-strain diagram is a straight line.

In the above definition of a stress, it has been assumed that the internal forces in the bar are uniformly distributed across the

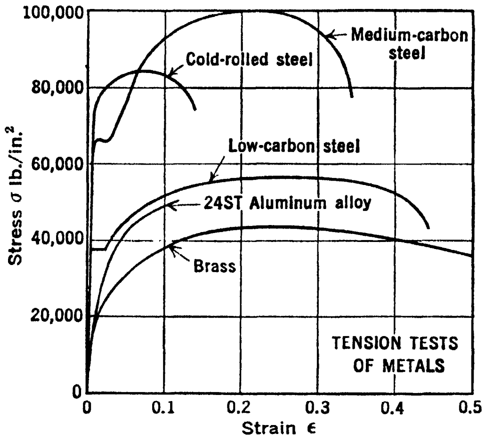


Fig. 7-2

normal section of the bar. In Fig. 7-3 is shown a free-body diagram of one-half of the bar of Fig. 7-1. In Fig. 7-3a the internal forces are shown as uniformly distributed over the section, while in (b) it is indicated that such internal forces might not be uniformly distributed. In case (b) the expression $\sigma = \frac{F}{A}$ would give only the average stress over the section, which might be quite different from the stress at any particular point. To define the stress at a given point consider a small element of area ΔA at that point on which a force ΔF acts. Then the stress at the point is $\sigma = \frac{\Delta F}{\Delta A}$ and this stress may vary from point to point. In what follows we shall always suppose that the stresses are uniformly distributed in the members.

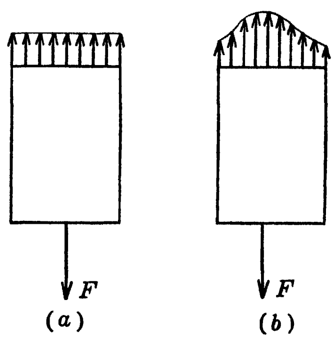


Fig. 7-3

Summarizing the assumptions which will underlie the following chapter, we have:

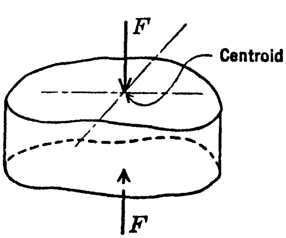
(1) The members are acted upon by axial tension or compression forces.
(2) The stresses are uniformly distributed.
(3) The properties of the material are the same for tension or compression.
(4) Hooke's Law applies.

The analysis of problems for which these assumptions do not apply is considered in books on Strength of Materials and Theory of Elasticity.

PROBLEMS

178. Find the relationship between the modulus of elasticity defined by Young in the quotation at the head of this chapter, and the present-day Young's Modulus E.

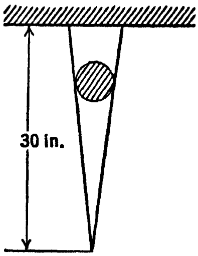
179. Show that a tension or compression force F, as shown in the figure, must be applied at the centroid of the area of the normal cross section in order that the stress be uniformly distributed across the area.



PROB. 179

180. A uniform bar of weight W, length l, and cross-section area A is attached at the top end to a rigid support and hangs vertically. Find the total extension of the bar due to gravity, and compare with the extension of a similar weightless bar with a concentrated load W at the end.

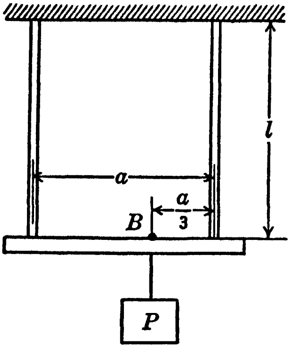
181. A conical bar of circular cross-section is suspended vertically from its base. The bar is made of steel. The upper radius of the bar is 1 in., and the length is 30 in. Find the total extension of the bar due to gravity forces.



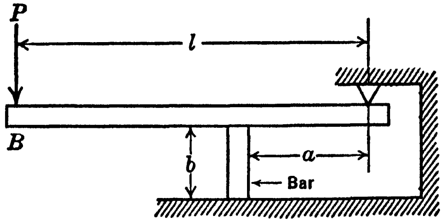
PROB. 181

182. Two vertical bars each of length l, cross-sectional area A, and modulus of elasticity E support a cross-piece upon which is hung a weight P as shown. What is the vertical displacement of point B on the cross-piece due to the elongation of the bars?

183. A beam carrying a load P is supported as shown in the diagram. The bar has a length b, a cross-sectional area A, and modulus of elasticity E. What is the vertical displacement of the point B due to the shortening of the bar? (Note: Make use of the fact that the displacement is very small compared to l.)

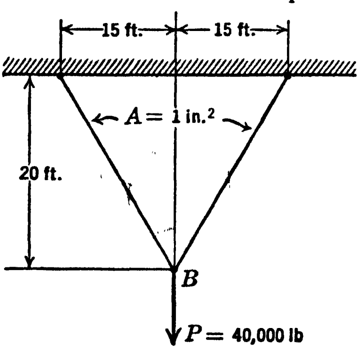


PROB. 182

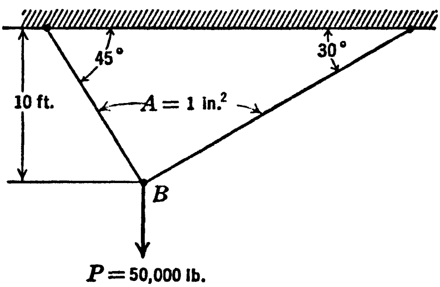


PROB. 183

184. A load $P = 40{,}000$ lb is supported by two steel bars, each of cross-sectional area 1 sq in. What is the vertical displacement of point B? (Note: The displacement of B is very small compared to the length of the bars.)



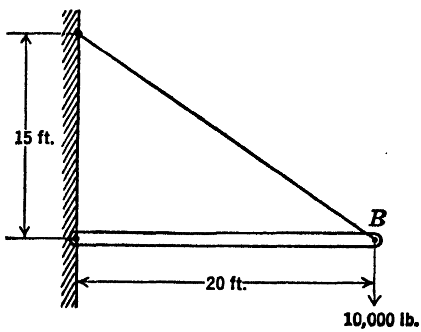
PROB. 184



PROB. 185

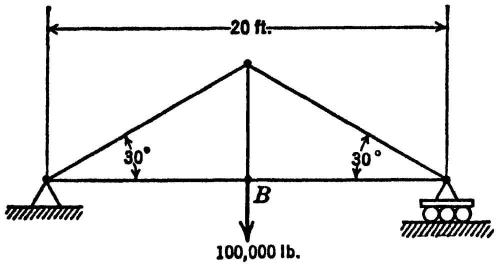
185. A load $P = 50{,}000$ pounds is supported by two steel bars each of cross-sectional area 1 sq in. What is the vertical displacement of point B? What is the horizontal displacement of point B?

186. A load of 10,000 pounds is supported as shown. The horizontal member is a 6 in. $\times$ 6 in. wood strut with $E = 1.6 \times 10^6$ lb per sq in.



PROB. 186

The inclined member is a $\frac{1}{2}$ sq in. steel rod. What is the vertical and horizontal displacement of point B?



PROB. 187

187. A load of 100,000 pounds is carried by a small truss as shown. Each member of the truss is steel and has a cross-sectional area of 10 sq in. What is the vertical displacement of point B?

Statically Indeterminate Problems in Tension and Compression. The methods to be used in treating a statically indeterminate problem can be most easily indicated by considering some specific examples.

Consider a structure composed of three vertical bars supporting a rigid cross-piece of negligible weight which carries a load P, as shown in Fig. 7–4. The bars are all straight and of uniform cross section, but the areas of the bars are different and the bars are made of different materials. In Fig. 7–4b is shown a free-body diagram of the cross-bar, with the three unknown forces F_1, F_2, and F_3. Since only two equations of statics can be written

$$\Sigma F_y = 0$$
$$\Sigma M = 0$$

there is one redundant constraint, so that the problem is statically indeterminate. From the condition of the problem that the crosspiece is rigid, we know that

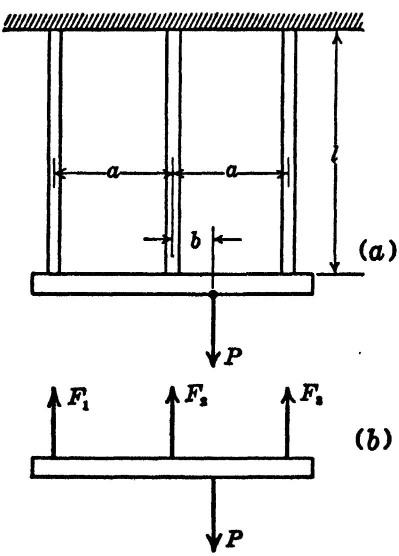


FIG. 7–4

the ends of the three bars must always lie along a straight line, so that

$$\delta_2 = \frac{\delta_1 + \delta_3}{2}$$

where δ_1, δ_2, and δ_3 are the total elongations of the three bars. Using Hooke's Law, these elongations can be written as functions of the forces, and we have:

$$\frac{F_2 l}{A_2 E_2} = \frac{1}{2}\left(\frac{F_1 l}{A_1 E_1} + \frac{F_3 l}{A_3 E_3}\right)$$

Adding the two equations of equilibrium, we have finally three equations from which the three unknown forces may be determined:

$$\begin{cases} F_1 + F_2 + F_3 - P = 0 \\ F_1 a + Pb - F_3 a = 0 \\ F_2 = \dfrac{A_2 E_2}{2}\left(\dfrac{F_1}{A_1 E_1} + \dfrac{F_3}{A_3 E_3}\right) \end{cases}$$

As a second example consider a framework consisting of three straight, weightless bars carrying a load P, as shown in Fig. 7–5a.

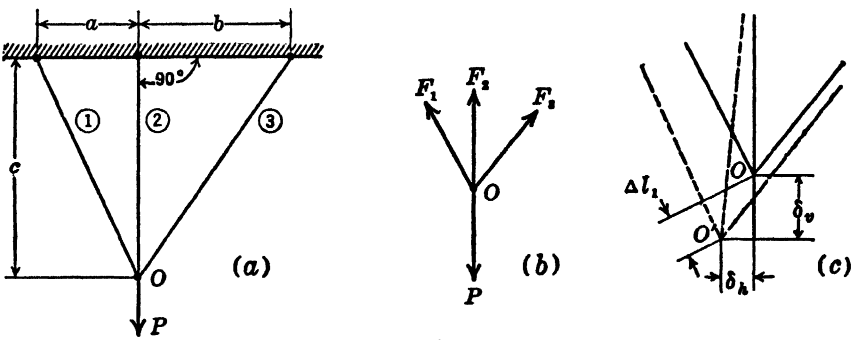


Fig. 7–5

The free-body diagram of the joint in Fig. 7–5b shows that there are three unknown forces to be determined, whereas only two equations of statics can be written. A third equation can be derived from the conditions of displacement of the system. When the load is applied to the framework, the joint will be displaced with a vertical component δ_v and a horizontal component δ_h as shown in Fig. 7–5c. The conditions of displacement, therefore, are that the joining ends of each of the three members must un-

dergo the same displacement. Assuming that these displacements are actually sufficiently small so that the members remain parallel to their original positions, these conditions may be expressed analytically as follows:

$$\delta_2 = \delta_v$$

$$\delta_1 = \frac{c}{l_1}\delta_v - \frac{a}{l_1}\delta_h$$

$$\delta_3 = \frac{c}{l_3}\delta_v + \frac{b}{l_3}\delta_h$$

where δ_1, δ_2, δ_3 are the total elongations of the three bars of length l_1, $l_2 = c$, and l_3. Eliminating the δ_v and the δ_h from these expressions, we have:

$$bl_1\delta_1 + al_3\delta_3 - c(a + b)\delta_2 = 0$$

which may be written in terms of the forces by using Hooke's Law:

$$\frac{bl_1^2F_1}{A_1E_1} + \frac{al_3^2F_3}{A_3E_3} - \frac{c(a + b)l_2F_2}{A_2E_2} = 0$$

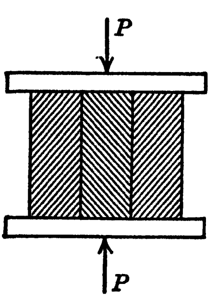
This equation, together with the two statics equations $\Sigma F_x = 0$, $\Sigma F_y = 0$, is sufficient to determine the unknown forces F_1, F_2, and F_3.

The preceding discussion can be summarized in the form of the following procedure for the solution of statically indeterminate problems:

(1) Determine the number of unknown forces N which are involved.

(2) Determine the number n of independent equations of equilibrium which can be written for the system.

(3) From the statement of the problem determine the $(N - n)$ conditions of compatibility for the displacements of the system.



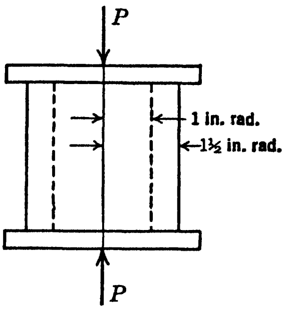
Prob. 188

PROBLEMS

188. Three identical bars are symmetrically loaded as is shown in the accompanying figure. The two outside bars are aluminum, and the inside bar is steel. The two bearing plates may be assumed to be rigid. Find the force in the steel bar in terms of the load P.

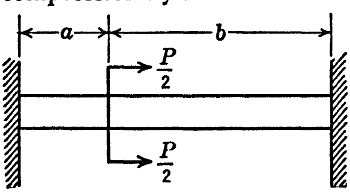
189. In the Example shown in Fig. 7–4, suppose that $a = 6$ in., $b = 2$ in., and that the two outside bars have cross-sectional areas of 1 sq in., while the middle bar has an area of 2 sq in. If all three bars are made of the same material, find the stresses in the bars when $P = 10{,}000$ lb.

190. In the Example shown in Fig. 7–5, $a = b = 2$ ft, the center bar is vertical, and the two outside bars make an angle of 30° with the center bar. The bars are made of steel, and all have the same cross-sectional area. Find the forces set up in the bars by a vertical load $P = 10{,}000$ lb.



PROB. 191

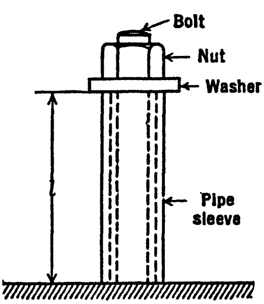
191. A tube of aluminum with a central cylinder of steel is loaded in compression by an axial load P as shown in the diagram. Find the stress in the aluminum when the load is such that the stress in the steel is 10,000 lb per sq in. What proportion of the load is carried by the steel cylinder?



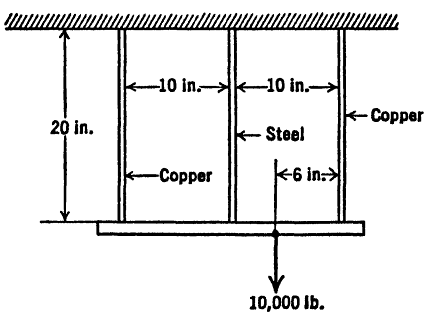
PROB. 192

192. A straight bar, rigidly built in at each end is loaded with an axial force P as shown in the diagram. What is the reaction at the left end of the bar?

193. A steel bolt of length l is rigidly built in at one end and is assembled with a steel pipe sleeve, washer and nut as shown. The cross-



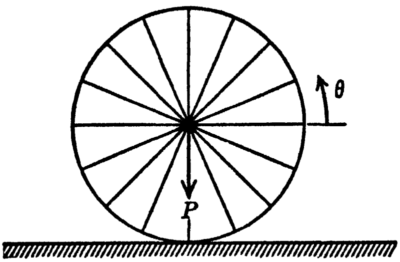
PROB. 193



PROB. 194

sectional area of the bolt is $\frac{1}{3}$ sq in., the cross-sectional area of the pipe is $\frac{1}{4}$ sq in., and the length l is 10 in. The pitch of the bolt thread is $\frac{1}{32}$ in. What stress is produced in the bolt by $\frac{1}{2}$ turn of the nut?

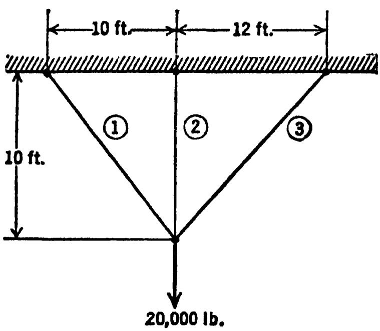
194. Two vertical copper bars and one vertical steel bar are assembled with a rigid cross-bar as shown. A 10,000-lb load is applied. What is the resulting stress in the steel bar? The cross-sectional area of the steel bar is 1 sq in. and the area of each copper bar is 2 sq in.



PROB. 195

195. A circular wheel has n uniformly spaced wire spokes each of length r and area A. The spokes all have an initial tension. A load P is then applied to the axle. What is the induced force in the spokes? Assume the rim is rigid and that n is large. (Hint: Since n is large, analyze a wheel that has $\frac{2\pi}{d\theta}$ spokes each of area $\frac{nA}{2\pi}\,d\theta$.)

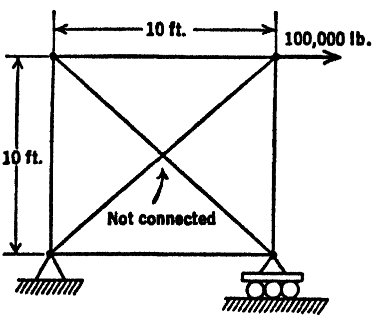
196. Three steel bars, each of area 1 sq in., carry a load of 20,000 lb as shown. What are the forces in the bars?



PROB. 196

197. A pin-connected frame is made of 6 steel members, each having a cross-sectional area of 5 sq in. A horizontal force of 100,000 lb is applied as shown. What forces are produced in the two diagonal members?

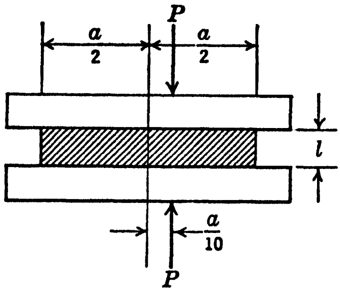
198. A square slab of material is compressed between two rigid plates as shown in the diagram. The elastic properties of the material are such that the pressure against the bearing plate is directly proportional to Δl at that point, i.e., $\sigma = k(\Delta l)$. What is the pressure distribution against the bearing plate if the load P is applied as shown?



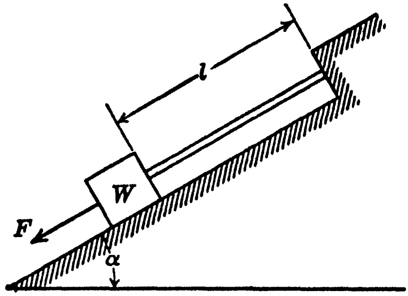
PROB. 197

199. A weight W is restrained from sliding down an inclined plane by a bar whose unstretched length is l. The coefficient of friction between W and the plane is μ. A force F is applied and W is pulled down the plane. F is then gradually reduced, permitting W to move up the plane until F is zero. The same procedure

is repeated except that the second time F pushes the weight up the plane and then permits it to slide down. If the modulus of elasticity



PROB. 198



PROB. 199

of the bar is E and the cross-sectional area A, where does the bar come to rest in these two cases?

Initial Stresses. Stresses may be set up in a statically indeterminate structure which may be maintained even if there are no external forces acting upon the structure. Such a structure is said to be *self-strained*. Such initial stresses are usually due either to inaccuracies of assembly or are thermal stresses. The way in which inaccuracies of assembly may set up initial stresses is shown in the following example.

EXAMPLE. Suppose that a tension member is to be made up of three eye-bars, pinned together at the ends as shown in Fig. 7–6a.

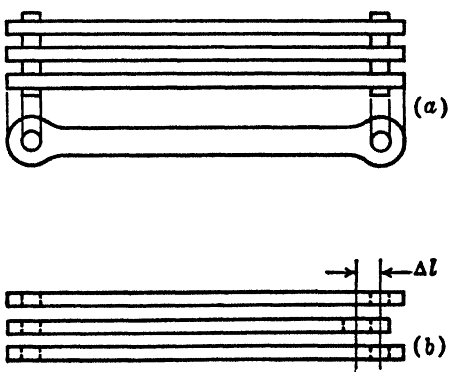


FIG. 7–6

We further suppose that due to inaccuracies in manufacture the center bar is shorter by an amount Δl than the two outer bars.

We then see that in order to insert the pins through the holes, the inside bar must be extended, and the two outside bars must be compressed, thus setting up initial stresses in the system. These stresses may be computed as follows.

Let δ_o be the total compression of the outside bars, and δ_i be the total extension of the inside bar. Then:

$$\delta_o + \delta_i = \Delta l$$

Further, the total force exerted on the pin by the inside bar, F_i, must be just equal in magnitude to the total force F_o exerted on the outside bars by the pin. Using Hooke's Law, and assuming that the bars have equal areas, this gives us:

$$\frac{F_o l}{2AE} + \frac{F_i l}{AE} = \Delta l$$

But $F_o = F_i$, so that:

$$F_o = F_i = \frac{2AE\,\Delta l}{3l}$$

Knowing the forces and the areas, the initial stresses may be determined.

PROBLEMS

200. In the preceding Example, take the cross-sectional area of the steel bars as 2 sq in., the length as 48 ft, and Δl as $\frac{1}{8}$ in. Find the initial stresses in the bars.

201. If the system of Problem 200 above is loaded with an axial tension load of 15,000 lb, find the stresses in the three bars.

202. Referring to Problem 190, suppose that initially the center bar is cut $\frac{1}{10}$ in. too short. Find the initial stresses set up in the system when the bars are connected together, if they have an area of 1 sq in.

203. Solve Problem 191, assuming that the center steel cylinder was accidentally made $\frac{1}{200}$ in. shorter than the outside aluminum tube. The length of the aluminum tube is 4 ft.

Thermal Stresses. Another way in which stresses can be set up in a statically indeterminate structure is by means of temperature changes. It is found experimentally that when the temperature of a bar of metal is increased, the length of the bar is also increased, and, conversely, a decrease in temperature is accompanied by a decrease in length. This change of length Δl_t is

found to be directly proportional to the length of the bar, and to the temperature difference:

$$\Delta l_t = \alpha l(t - t_o)$$

The coefficient of thermal expansion α must be determined experimentally. It has different values for different materials as shown in the following table:

COEFFICIENT OF THERMAL EXPANSION

Material	α *(in./in.°F)*
Steel	6.5×10^{-6}
Copper	9.3×10^{-6}
Aluminum	12.8×10^{-6}
Brass	10.4×10^{-6}
Tungsten	2.7×10^{-6}

If the bar is free to expand or to contract during temperature changes, the length will vary without setting up stresses in the bar. If the bar is constrained, however, so that part or all of the expansion is prevented, stresses will be set up. Such stresses are called *thermal stresses*. Suppose, for example, that a bar is completely constrained. Then, after a temperature increase, the bar will have in it a compression strain just equal to the thermal expansion strain which the bar would have had if it had been free to expand. The compressive thermal stress is thus:

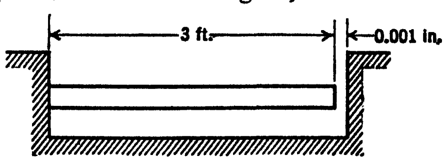
$$\sigma_t = E\epsilon_t = E\frac{\alpha l(t - t_o)}{l}$$
$$= E\alpha(t - t_o)$$

There will be a force in the bar corresponding to this stress:

$$\sigma_t = \frac{F_t}{A}; \quad F_t = AE\alpha(t - t_o)$$

PROBLEMS

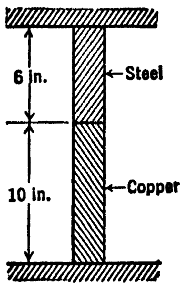
204. A 3-ft long copper bar having a 1-in. diameter circular cross section is arranged as shown in the figure, with a 0.001-in. gap between



PROB. 204

its end and a rigid wall at room temperature. If the temperature increases 60° F, find the stress in the rod.

205. If the framework of Problem 190 were assembled with no initial stresses at a temperature of 70° F, find the thermal stresses in the system when the temperature drops to 20° F. The bars have an area of 1 sq in., and no load acts on the frame.

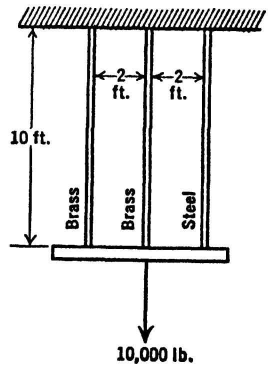


PROB. 206

206. A steel bar and a copper bar each of 4 sq in. area are restrained between two rigid supports as shown. If the temperature is increased 100°, what stresses are induced?

207. A steel train rail is installed in a continuous piece without expansion joints. If the temperature falls 100° below installation temperature, what stress is induced?

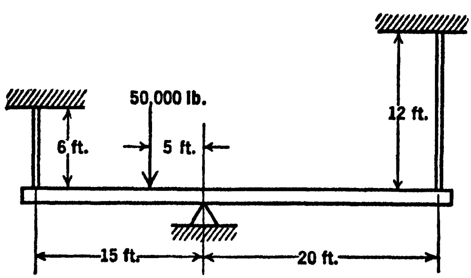
208. Three bars which are initially the same length support a rigid cross-piece upon which is carried a load of 10,000 lb as shown. The temperature is increased 100°. What is the final stress in the steel bar? Each bar has an area of 1 sq in.



PROB. 208

209. A rigid beam rests on one interior support and is supported at each end by a steel bar of 2 sq in. area. A load of 50,000 lb is applied as shown. What stresses are produced in the bars? If the temperature then increases 80°, what stresses are in the bars?

210. A steel rim is to be installed on a locomotive wheel. When the wheel is at 75° and the rim is at 200° the inside diameter of the rim is 0.02 in. larger than the outside diameter of the wheel. When the assembly cools to 75°, what circumferential tension stress will there be in the rim? The outside



PROB. 209

diameter of the wheel is 5 ft 0 in. Assume the rim has a rectangular cross section 1 in. × 3 in. Treat the locomotive wheel as being rigid.

211. If the circumferential tension stress is 10,000 lb per sq in. in Problem 210, what is the pressure exerted by the rim on the wheel? (Draw a free-body diagram of one-half of the rim.)

APPENDIX I

BIBLIOGRAPHY

The subjects discussed in this book are treated in a very large number of books. The bibliography below includes only a selection of those books which present the subject from essentially the same point of view as this book. These references may thus be of assistance to the student who desires additional explanations, illustrative examples, and problems.

1. Brown, F. L., *Engineering Mechanics*, 2nd Edition, John Wiley & Sons, New York, 1942.
2. Daugherty, R. L., *Hydraulics*, 4th Edition, McGraw-Hill Book Co., New York, 1937.
3. Faires, V. M., *Design of Machine Elements*, The Macmillan Co., New York, 1936.
4. Jeans, J. H., *Theoretical Mechanics*, Ginn & Co., Boston, 1935.
5. Karman, Th., and Biot, M. A., *Mathematical Methods in Engineering*, McGraw-Hill Book Co., 1940.
6. Lamb, H., *Statics*, Cambridge University Press, 1946.
7. Mach, E., *The Science of Mechanics*, Open Court Publishing Co., Chicago, 1893.
8. Millikan, R. A., Roller, D., and Watson, E. C., *Mechanics, Molecular Physics, Heat and Sound*, Ginn & Co., Boston, 1937.
9. Norton, A. E., *Lubrication*, McGraw-Hill Book Co., New York, 1942.
10. Prandtl, L., and Tietjens, O. G., *Fundamentals of Hydro- and Aeromechanics*, McGraw-Hill Book Co., New York, 1934.
11. Sutherland, H., and Bowman, H. L., *Structural Theory*, 3rd Edition, John Wiley & Sons, New York, 1942.
12. Timoshenko, S. P., *Strength of Materials*, Vol. I, 2nd Edition, D. Van Nostrand Company, New York, 1940.
13. Timoshenko, S. P., and MacCullough, G. H., *Elements of Strength of Materials*, 3rd Edition, D. Van Nostrand Co., New York, 1949.

14. Timoshenko, S. P., and Young, D. H., *Engineering Mechanics, Statics*, McGraw-Hill Book Co., New York, 1937.
15. Timoshenko, S. P., and Young, D. H., *Theory of Structures*, McGraw-Hill Book Co., New York, 1945.

Specific chapter references follow:

Chapter I: Ref. 1, 4, 5, 6, 14
Chapter II: Ref. 1, 14
Chapter III: Ref. 4, 5, 14
Chapter IV: Ref. 1, 2, 6, 10, 14
Chapter V: Ref. 1, 3, 9, 14
Chapter VI: Ref. 6, 11, 12, 13, 14, 15
Chapter VII: Ref. 12, 13

APPENDIX II

PROPERTIES OF PLANE SECTIONS

The following symbols will be used:

A = Area
x_c, y_c = Coordinates of centroid of section in xy coordinate system.
I_{x_c}, I_{y_c} = Moment of inertia about an axis through the centroid parallel to the xy axes.
r_{x_c}, r_{y_c} = Radius of gyration of the section with respect to the centroidal axes parallel to the xy axes.
$I_{x_c y_c}$ = Product of inertia with respect to the centroidal axes parallel to the xy axes.
I_x, I_y = Moment of inertia with respect to the xy axes shown.
r_x, r_y = Radius of gyration of the section with respect to the xy axes shown.
I_{xy} = Product of inertia with respect to the xy axes shown.
I_P = Polar moment of inertia about an axis passing through the centroid.
r_P = Radius of gyration of the section about the polar axis passing through the centroid.

G marks the centroid.

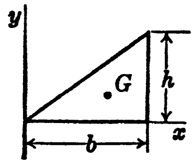
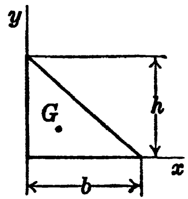
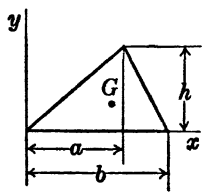
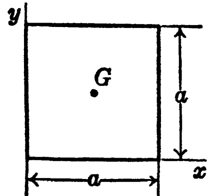
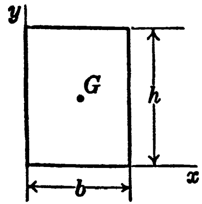
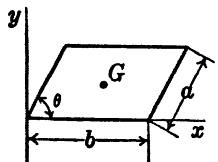
Figure	Area and Centroid	Moment of Inertia	r^2	Product of Inertia
1  Right Triangle	$A=\frac{1}{2}bh$ $x_c=\frac{2}{3}b$ $y_c=\frac{1}{3}h$	$I_{x_c}=\frac{bh^3}{36}$ $I_{y_c}=\frac{b^3h}{36}$ $I_x=\frac{bh^3}{12}$ $I_y=\frac{b^3h}{4}$	$r_{x_c}^2=\frac{1}{18}h^2$ $r_{y_c}^2=\frac{1}{18}b^2$ $r_x^2=\frac{1}{6}h^2$ $r_y^2=\frac{1}{2}b^2$	$I_{x_cy_c}=\frac{A}{36}hb=\frac{h^2b^2}{72}$ $I_{xy}=\frac{A}{4}hb=\frac{h^2b^2}{8}$
2 	$A=\frac{1}{2}bh$ $x_c=\frac{1}{3}b$ $y_c=\frac{1}{3}h$	$I_{x_c}=\frac{bh^3}{36}$ $I_{y_c}=\frac{b^3h}{36}$ $I_x=\frac{bh^3}{12}$ $I_y=\frac{b^3h}{12}$	$r_{x_c}^2=\frac{1}{18}h^2$ $r_{y_c}^2=\frac{1}{18}b^2$ $r_x^2=\frac{1}{6}h^2$ $r_y^2=\frac{1}{6}b^2$	$I_{x_cy_c}=-\frac{A}{36}hb=-\frac{h^2b^2}{72}$ $I_{xy}=\frac{A}{12}hb=\frac{h^2b^2}{24}$
3  Triangle	$A=\frac{1}{2}bh$ $x_c=\frac{1}{3}(a+b)$ $y_c=\frac{1}{3}h$	$I_{x_c}=\frac{bh^3}{36}$ $I_{y_c}=\frac{bh}{36}(b^2-ab+a^2)$ $I_x=\frac{bh^3}{12}$ $I_y=\frac{bh}{12}(b^2+ab+a^2)$	$r_{x_c}^2=\frac{1}{18}h^2$ $r_{y_c}^2=\frac{1}{18}(b^2-ab+a^2)$ $r_x^2=\frac{1}{6}h^2$ $r_y^2=\frac{1}{6}(b^2+ab+a^2)$	$I_{x_cy_c}=\frac{Ah}{36}(2a-b)=\frac{bh^2}{72}(2a-b)$ $I_{xy}=\frac{Ah}{12}(2a+b)=\frac{bh^2}{24}(2a+b)$

	Figure	Area and Centroid	Moment of Inertia	r^2	Product of Inertia
4	 Square	$A = a^2$ $x_c = \frac{1}{2}a$ $y_c = \frac{1}{2}a$	$I_{x_c} = I_{y_c} = \frac{A}{12}a^2$ $I_x = I_y = \frac{A}{3}a^2$ $I_P = \frac{A}{6}a^2$	$r_{x_c}^2 = r_{y_c}^2 = \frac{1}{12}a^2$ $r_x^2 = r_y^2 = \frac{1}{3}a^2$ $r_P^2 = \frac{1}{6}a^2$	$I_{x_cy_c} = 0$ $I_{xy} = \frac{A}{4}a^2 = \frac{a^4}{4}$
5	 Rectangle	$A = bh$ $x_c = \frac{1}{2}b$ $y_c = \frac{1}{2}h$	$I_{x_c} = \frac{bh^3}{12}$ $I_{y_c} = \frac{b^3h}{12}$ $I_x = \frac{bh^3}{3}$ $I_y = \frac{b^3h}{3}$ $I_P = \frac{bh}{12}(b^2+h^2)$	$r_{x_c}^2 = \frac{1}{12}h^2$ $r_{y_c}^2 = \frac{1}{12}b^2$ $r_x^2 = \frac{1}{3}h^2$ $r_y^2 = \frac{1}{3}b^2$ $r_P^2 = \frac{1}{12}(b^2+h^2)$	$I_{x_cy_c} = 0$ $I_{xy} = \frac{A}{4}bh = \frac{b^2h^2}{4}$
6	 Parallelogram	$A = ab \sin\theta$ $x_c = \frac{1}{2}(b + a\cos\theta)$ $y_c = \frac{1}{2}(a\sin\theta)$	$I_{x_c} = \frac{a^3b}{12}\sin^3\theta$ $I_{y_c} = \frac{ab}{12}\sin\theta\,(b^2+a^2\cos^2\theta)$ $I_x = \frac{a^3b}{3}\sin^3\theta$ $I_y = \frac{ab}{3}\sin\theta\,(b+a\cos\theta)^2$ $- \frac{a^2b^2}{6}\sin\theta\cos\theta$	$r_{x_c}^2 = \frac{1}{12}(a\sin\theta)^2$ $r_{y_c}^2 = \frac{1}{12}(b^2+a^2\cos^2\theta)$ $r_x^2 = \frac{1}{3}(a\sin\theta)^2$ $r_y^2 = \frac{1}{3}(b+a\cos\theta)^2$ $-\frac{1}{6}(ab\cos\theta)$	$I_{x_cy_c} = \frac{a^3b}{12}\sin^2\theta\cos\theta$

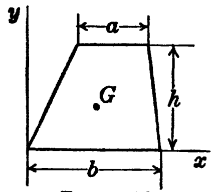
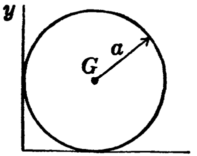
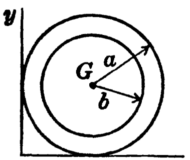
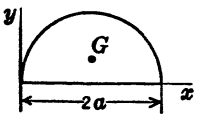
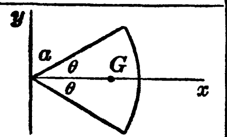
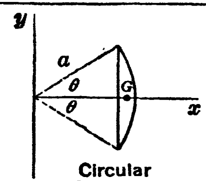
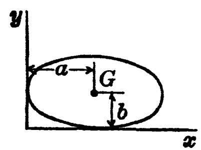
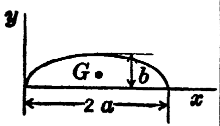
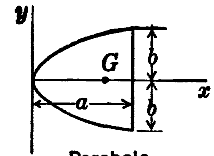
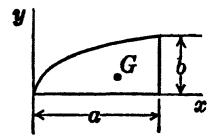
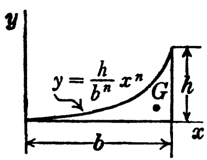
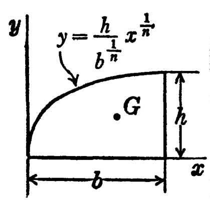
7	 Trapezoid	$A = \frac{1}{2}h(a+b)$ $y_c = \frac{1}{3}h\left(\frac{2a+b}{a+b}\right)$	$I_{x_c} = \frac{h^3(a^2+4ab+b^2)}{36(a+b)}$ $I_x = \frac{h^3(3a+b)}{12}$	${r_{x_c}}^2 = \frac{h^2(a^2+4ab+b^2)}{18(a+b)^2}$ ${r_x}^2 = \frac{h^2(3a+b)}{6(a+b)}$	
8	 Circle	$A = \pi a^2$ $x_c = a$ $y_c = a$	$I_{x_c} = \frac{1}{4}\pi a^4 = I_{y_c}$ $I_x = I_y = \frac{5}{4}\pi a^4$ $I_P = \frac{1}{2}\pi a^4$	${r_{x_c}}^2 = {r_{y_c}}^2 = \frac{1}{4}a^2$ ${r_x}^2 = {r_y}^2 = \frac{5}{4}a^2$ ${r_P}^2 = \frac{1}{2}a^2$	$I_{x_c y_c} = 0$ $I_{xy} = Aa^2$
9	 Annulus	$A = \pi(a^2 - b^2)$ $x_c = a$ $y_c = a$	$I_{x_c} = I_{y_c} = \frac{\pi}{4}(a^4 - b^4)$ $I_x = I_y = \frac{5}{4}\pi a^4 - \pi a^2 b^2 - \frac{\pi}{4}b^4$ $I_P = \frac{\pi}{2}(a^4 - b^4)$	${r_{x_c}}^2 = {r_{y_c}}^2 = \frac{1}{4}(a^2+b^2)$ ${r_x}^2 = {r_y}^2 = \frac{1}{4}(5a^2+b^2)$ ${r_P}^2 = \frac{1}{2}(a^2+b^2)$	$I_{x_c y_c} = 0$ $I_{xy} = Aa^2 = \pi a^2(a^2 - b^2)$
10	 Semi-circle	$A = \frac{1}{2}\pi a^2$ $x_c = a$ $y_c = \frac{4a}{3\pi}$	$I_{x_c} = \frac{a^4(9\pi^2 - 64)}{72\pi}$ $I_{y_c} = \frac{1}{8}\pi a^4$ $I_x = \frac{1}{8}\pi a^4$ $I_y = \frac{5}{8}\pi a^4$	${r_{x_c}}^2 = \frac{a^2(9\pi^2 - 64)}{36\pi^2}$ ${r_{y_c}}^2 = \frac{1}{4}a^2$ ${r_x}^2 = \frac{1}{4}a^2$ ${r_y}^2 = \frac{5}{4}a^2$	$I_{x_c y_c} = 0$ $I_{xy} = \frac{2}{3}a^4$

	Figure	Area and Centroid	Moment of Inertia	r^2	Product of Inertia
11	 Circular Sector	$A = a^2\theta$ $x_c = \frac{2a}{3}\frac{\sin\theta}{\theta}$ $y_c = 0$	$I_x = \frac{1}{4}a^4(\theta - \sin\theta\cos\theta)$ $I_y = \frac{1}{4}a^4(\theta + \sin\theta\cos\theta)$	$r_x^2 = \frac{1}{4}a^2\left(\frac{\theta - \sin\theta\cos\theta}{\theta}\right)$ $r_y^2 = \frac{1}{4}a^2\left(\frac{\theta + \sin\theta\cos\theta}{\theta}\right)$	$I_{x_c y_c} = 0$ $I_{xy} = 0$
12	 Circular Segment	$A = a^2(\theta - \frac{1}{2}\sin 2\theta)$ $x_c = \frac{2a}{3}\left(\frac{\sin^3\theta}{\theta - \sin\theta\cos\theta}\right)$ $y_c = 0$	$I_x = \frac{Aa^2}{4}\left[1 - \frac{2\sin^3\theta\cos\theta}{3(\theta - \sin\theta\cos\theta)}\right]$ $I_y = \frac{Aa^2}{4}\left[1 + \frac{2\sin^3\theta\cos\theta}{\theta - \sin\theta\cos\theta}\right]$	$r_x^2 = \frac{a^2}{4}\left[1 - \frac{2\sin^3\theta\cos\theta}{3(\theta - \sin\theta\cos\theta)}\right]$ $r_y^2 = \frac{a^2}{4}\left[1 + \frac{2\sin^3\theta\cos\theta}{\theta - \sin\theta\cos\theta}\right]$	$I_{x_c y_c} = 0$ $I_{xy} = 0$
13	 Ellipse	$A = \pi ab$ $x_c = a$ $y_c = b$	$I_{x_c} = \frac{\pi}{4}ab^3$ $I_{y_c} = \frac{\pi}{4}a^3b$ $I_x = \frac{5}{4}\pi ab^3$ $I_y = \frac{5}{4}\pi a^3b$ $I_P = \frac{\pi ab}{4}(a^2+b^2)$	$r_{x_c}^2 = \frac{1}{4}b^2$ $r_{y_c}^2 = \frac{1}{4}a^2$ $r_x^2 = \frac{5}{4}b^2$ $r_y^2 = \frac{5}{4}a^2$ $r_P^2 = \frac{1}{4}(a^2 + b^2)$	$I_{x_c y_c} = 0$ $I_{xy} = Aab = \pi a^2b^2$
14	 Semi-ellipse	$A = \frac{1}{2}\pi ab$ $x_c = a$ $y_c = \frac{4b}{3\pi}$	$I_{x_c} = \frac{ab^3}{72\pi}(9\pi^2 - 64)$ $I_{y_c} = \frac{\pi}{8}a^3b$ $I_x = \frac{\pi}{8}ab^3$ $I_y = \frac{5}{8}\pi a^3b$	$r_{x_c}^2 = \frac{b^2}{36\pi^2}(9\pi^2 - 64)$ $r_{y_c}^2 = \frac{1}{4}a^2$ $r_x^2 = \frac{1}{4}b^2$ $r_y^2 = \frac{5}{4}a^2$	$I_{x_c y_c} = 0$ $I_{xy} = \frac{2}{3}a^2b^2$

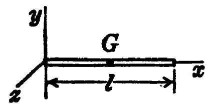
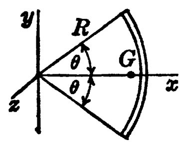
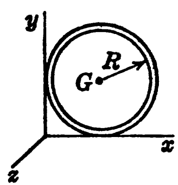
15	 Parabola	$A = \frac{4}{3}ab$ $x_c = \frac{3}{5}a$ $y_c = 0$	$I_{x_c} = I_x = \frac{4}{15}ab^3$ $I_{y_c} = \frac{16}{175}a^3b$ $I_y = \frac{4}{7}a^3b$	$r_{x_c}^2 = r_x^2 = \frac{1}{5}b^2$ $r_{y_c}^2 = \frac{12}{175}a^2$ $r_y^2 = \frac{3}{7}a^2$	$I_{x_c y_c} = 0$ $I_{xy} = 0$
16	 Semi-parabola	$A = \frac{2}{3}ab$ $x_c = \frac{3}{5}a$ $y_c = \frac{3}{8}b$	$I_x = \frac{2}{15}ab^3$ $I_y = \frac{2}{7}a^3b$	$r_x^2 = \frac{1}{5}b^2$ $r_y^2 = \frac{3}{7}a^2$	$I_{xy} = \frac{A}{4}ab = \frac{1}{6}a^2b^2$
17	 n^{th} degree parabola	$A = \frac{bh}{n+1}$ $x_c = \frac{n+1}{n+2}b$ $y_c = \frac{h}{2}\left(\frac{n+1}{2n+1}\right)$	$I_x = \frac{bh^3}{3(3n+1)}$ $I_y = \frac{hb^3}{n+3}$	$r_x^2 = \frac{h^2(n+1)}{3(3n+1)}$ $r_y^2 = \frac{n+1}{n+3}b^2$	
18	 n^{th} degree parabola	$A = \frac{n}{n+1}bh$ $x_c = \frac{n+1}{2n+1}b$ $y_c = \frac{n+1}{2(n+2)}h$	$I_x = \frac{n}{3(n+3)}bh^3$ $I_y = \frac{n}{3n+1}hb^3$	$r_x^2 = \frac{n+1}{3(n+3)}h^2$ $r_y^2 = \frac{n+1}{3n+1}b^2$	

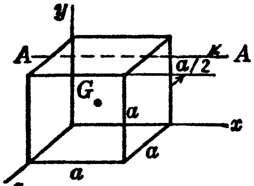
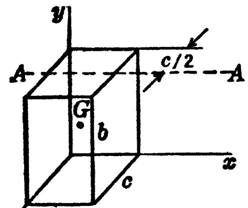
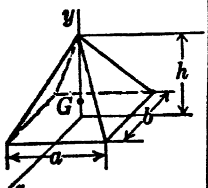
PROPERTIES OF HOMOGENEOUS BODIES

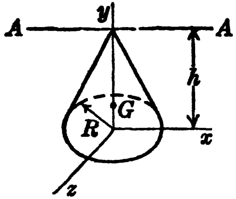
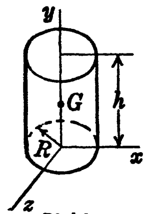
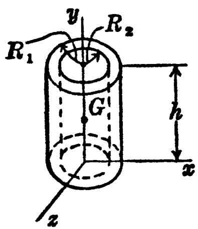
The following symbols will be used:

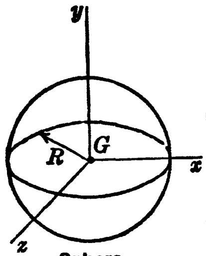
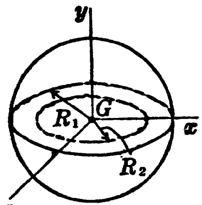
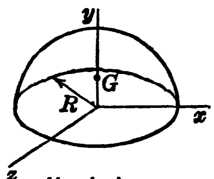
ρ = Mass density
M = Mass
x_c, y_c, z_c = Coordinates of centroid in xyz coordinate system.
$I_{x_c}, I_{y_c}, I_{z_c}$ = Moment of inertia about an axis through the centroid parallel to the xyz axes shown.
$r_{x_c}, r_{y_c}, r_{z_c}$ = Radius of gyration of the body with respect to the centroidal axes parallel to the xyz axes shown.
$I_{x_c y_c}, I_{x_c z_c}$, etc. = Product of inertia with respect to the centroidal axes parallel to the xyz axes shown.
I_x, I_y, I_z = Moment of inertia with respect to the xyz axes shown.
r_x, r_y, r_z = Radius of gyration of the body with respect to the xyz axes shown.
I_{xy}, I_{xz}, etc. = Product of inertia with respect to the xyz axes shown.
I_{AA}, r_{AA} = Moments of inertia and radii of gyration with respect to special axes shown.

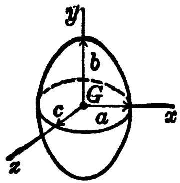
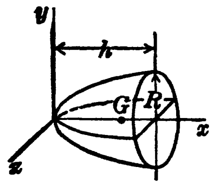
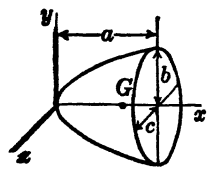
G marks the centroid

	Body	Mass and Centroid	Moment of Inertia	r^2	Product of Inertia
19	 Thin Rod	$M = \rho l$ $x_c = \frac{1}{2}l$ $y_c = 0$ $z_c = 0$	$I_x = I_{x_c} = 0$ $I_{y_c} = I_{z_c} = \frac{M}{12} l^2$ $I_y = I_z = \frac{M}{3} l^2$	$r_x^2 = r_{x_c}^2 = 0$ $r_{y_c}^2 = r_{z_c}^2 = \frac{1}{12}l^2$ $r_y^2 = r_z^2 = \frac{1}{3}l^2$	$I_{x_c y_c}$, etc. $= 0$ I_{xy}, etc. $= 0$
20	 Thin Circular Rod	$M = 2\rho R\theta$ $x_c = \frac{R \sin\theta}{\theta}$ $y_c = 0$ $z_c = 0$	$I_x = I_{x_c}$ $= \frac{MR^2(\theta - \sin\theta\cos\theta)}{2\theta}$ $I_y = \frac{MR^2(\theta + \sin\theta\cos\theta)}{2\theta}$ $I_z = MR^2$	$r_x^2 = r_{x_c}^2 = \frac{R^2(\theta - \sin\theta\cos\theta)}{2\theta}$ $r_y^2 = \frac{R^2(\theta + \sin\theta\cos\theta)}{2\theta}$ $r_z^2 = R^2$	$I_{x_c y_c}$, etc. $= 0$ I_{xy}, etc. $= 0$
21	 Thin Hoop	$M = 2\pi\rho R$ $x_c = R$ $y_c = R$ $z_c = 0$	$I_{x_c} = I_{y_c} = \frac{M}{2} R^2$ $I_{z_c} = MR^2$ $I_x = I_y = \frac{3}{2}MR^2$ $I_z = 3MR^2$	$r_{x_c}^2 = r_{y_c}^2 = \frac{1}{2}R^2$ $r_{z_c}^2 = R^2$ $r_x^2 = r_y^2 = \frac{3}{2}R^2$ $r_z = 3R^2$	$I_{x_c y_c}$, etc. $= 0$ $I_{xy} = MR^2$ $I_{xz} = I_{yz} = 0$

	Body	Mass and Centroid	Moment of Inertia	r^2	Product of Inertia
22	 Cube	$M = \rho a^3$ $x_c = \frac{1}{2}a$ $y_c = \frac{1}{2}a$ $z_c = \frac{1}{2}a$	$I_{x_c} = I_{y_c} = I_{z_c} = \frac{1}{6}Ma^2$ $I_x = I_y = I_z = \frac{2}{3}Ma^2$ $I_{AA} = \frac{5}{12}Ma^2$	$r_{x_c}^2 = r_{y_c}^2 = r_{z_c}^2 = \frac{1}{6}a^2$ $r_x^2 = r_y^2 = r_z^2 = \frac{2}{3}a^2$ $r_{AA}^2 = \frac{5}{12}a^2$	$I_{x_c y_c}$, etc. $= 0$ I_{xy}, etc. $= \frac{1}{4}Ma^2$
23	 Rectangular Prism	$M = \rho abc$ $x_c = \frac{1}{2}a$ $y_c = \frac{1}{2}b$ $z_c = \frac{1}{2}c$	$I_{x_c} = \frac{1}{12}M(b^2 + c^2)$ $I_x = \frac{1}{3}M(b^2 + c^2)$ $I_{AA} = \frac{1}{12}M(4b^2 + c^2)$	$r_{x_c}^2 = \frac{1}{12}(b^2 + c^2)$ $r_x^2 = \frac{1}{3}(b^2 + c^2)$ $r_{AA}^2 = \frac{1}{12}(4b^2 + c^2)$	$I_{x_c y_c}$, etc. $= 0$ $I_{xy} = \frac{1}{4}Mab$ $I_{xz} = \frac{1}{4}Mac$ $I_{yz} = \frac{1}{4}Mbc$
24	 Right Rectangular Pyramid	$M = \frac{1}{3}\rho abh$ $x_c = 0$ $y_c = \frac{1}{4}h$ $z_c = 0$	$I_{x_c} = \frac{1}{80}M(4b^2 + 3h^2)$ $I_{y_c} = I_y = \frac{1}{20}M(a^2 + b^2)$ $I_x = \frac{1}{20}M(b^2 + 2h^2)$ $I_z = \frac{1}{20}M(a^2 + 2h^2)$	$r_{x_c}^2 = \frac{1}{80}(4b^2 + 3h^2)$ $r_{y_c}^2 = r_y^2 = \frac{1}{20}(a^2 + b^2)$ $r_x^2 = \frac{1}{20}(b^2 + 2h^2)$ $r_z^2 = \frac{1}{20}(a^2 + 2h^2)$	$I_{x_c y_c}$, etc. $= 0$ I_{xy}, etc. $= 0$

25  Right Circular Cone	$M = \frac{1}{3}\pi\rho R^2 h$ $x_c = 0$ $y_c = \frac{1}{4}h$ $z_c = 0$	$I_{x_c} = I_{z_c} = \frac{3M}{80}(4R^2 + h^2)$ $I_{y_c} = I_y = \frac{3}{10}MR^2$ $I_x = I_z = \frac{1}{20}M(3R^2 + 2h^2)$ $I_{AA} = \frac{3}{20}M(R^2 + 4h^2)$	$r_{x_c}^2 = r_{z_c}^2 = \frac{3}{80}(4R^2 + h^2)$ $r_{y_c}^2 = \frac{3}{10}R^2$ $r_x^2 = r_z^2 = \frac{1}{20}(3R^2 + 2h^2)$ $r_{AA}^2 = \frac{3}{20}(R^2 + 4h^2)$	$I_{x_cy_c}$, etc. $= 0$ I_{xy}, etc. $= 0$
26  Right Circular Cylinder	$M = \pi\rho R^2 h$ $x_c = 0$ $y_c = \frac{1}{2}h$ $z_c = 0$	$I_{x_c} = I_{z_c} = \frac{1}{12}M(3R^2 + h^2)$ $I_{y_c} = I_y = \frac{1}{2}MR^2$ $I_x = I_z = \frac{1}{12}M(3R^2 + 4h^2)$	$r_{x_c}^2 = r_{z_c}^2 = \frac{1}{12}(3R^2 + h^2)$ $r_{y_c}^2 = r_y^2 = \frac{1}{2}R^2$ $r_x^2 = r_z^2 = \frac{1}{12}(3R^2 + 4h^2)$	$I_{x_cy_c}$, etc. $= 0$ I_{xy}, etc. $= 0$
27  Hollow Right Circular Cylinder	$M = \pi\rho h(R_1^2 - R_2^2)$ $x_c = 0$ $y_c = \frac{1}{2}h$ $z_c = 0$	$I_{x_c} = I_{z_c}$ $= \frac{1}{12}M(3R_1^2 + 3R_2^2 + h^2)$ $I_{y_c} = I_y = \frac{1}{2}M(R_1^2 + R_2^2)$ $I_x = I_z$ $= \frac{1}{12}M(3R_1^2 + 3R_2^2 + 4h^2)$	$r_{x_c}^2 = r_{z_c}^2 = \frac{1}{12}(3R_1^2 + 3R_2^2 + h^2)$ $r_{y_c}^2 = r_y^2 = \frac{1}{2}(R_1^2 + R_2^2)$ $r_x^2 = r_z^2 = \frac{1}{12}(3R_1^2 + 3R_2^2 + 4h^2)$	$I_{x_cy_c}$, etc. $= 0$ I_{xy}, etc. $= 0$

Body	Mass and Centroid	Moment of Inertia	r^2	Product of Inertia
28  Sphere	$M = \frac{4}{3}\pi\rho R^3$ $x_c = 0$ $y_c = 0$ $z_c = 0$	$I_{x_c} = I_x = \frac{2}{5}MR^2$ $I_{y_c} = I_y = \frac{2}{5}MR^2$ $I_{z_c} = I_z = \frac{2}{5}MR^2$	${r_{x_c}}^2 = {r_x}^2 = \frac{2}{5}R^2$ ${r_{y_c}}^2 = {r_y}^2 = \frac{2}{5}R^2$ ${r_{z_c}}^2 = {r_z}^2 = \frac{2}{5}R^2$	$I_{x_c y_c}$, etc. $= 0$
29  Hollow Sphere	$M = \frac{4}{3}\pi\rho(R_1^3 - R_2^3)$ $x_c = 0$ $y_c = 0$ $z_c = 0$	$I_x = I_y = I_z$ $= \frac{2}{5} M \frac{R_1^5 - R_2^5}{R_1^3 - R_2^3}$	${r_x}^2 = {r_y}^2 = {r_z}^2$ $= \frac{2}{5} \frac{R_1^5 - R_2^5}{R_1^3 - R_2^3}$	I_{xy}, etc. $= 0$
30  Hemisphere	$M = \frac{2}{3}\pi\rho R^3$ $x_c = 0$ $y_c = \frac{3}{8}R$ $z_c = 0$	$I_x = I_y = I_z = \frac{2}{5}MR^2$	${r_x}^2 = {r_y}^2 = {r_z}^2 = \frac{2}{5}R^2$	$I_{x_c y_c}$, etc. $= 0$ I_{xy}, etc. $= 0$

31  Ellipsoid	$M = \frac{4}{3}\pi\rho abc$ $x_c = 0$ $y_c = 0$ $z_c = 0$	$I_x = \frac{1}{5}M(b^2 + c^2)$ $I_y = \frac{1}{5}M(a^2 + c^2)$ $I_z = \frac{1}{5}M(a^2 + b^2)$	$r_x^2 = \frac{1}{5}(b^2 + c^2)$ $r_y^2 = \frac{1}{5}(a^2 + c^2)$ $r_z^2 = \frac{1}{5}(a^2 + b^2)$	I_{xy}, etc. $= 0$
32  Paraboloid of Revolution	$M = \frac{1}{2}\pi\rho R^2 h$ $x_c = \frac{2}{3}h$ $y_c = 0$ $z_c = 0$	$I_{x_c} = I_x = \frac{1}{3}MR^2$ $I_{y_c} = I_{z_c} = \frac{1}{18}M(3R^2 + h^2)$ $I_y = I_z = \frac{1}{6}M(R^2 + 3h^2)$	$r_{x_c}^2 = r_x^2 = \frac{1}{3}R^2$ $r_{y_c}^2 = r_{z_c}^2 = \frac{1}{18}(3R^2 + h^2)$ $r_y^2 = r_z^2 = \frac{1}{6}(R^2 + 3h^2)$	$I_{x_c y_c}$, etc. $= 0$ I_{xy}, etc. $= 0$
33  Elliptic Paraboloid	$M = \frac{1}{2}\pi\rho abc$ $x_c = \frac{2}{3}a$ $y_c = 0$ $z_c = 0$	$I_{x_c} = I_x = \frac{1}{6}M(b^2 + c^2)$ $I_{y_c} = \frac{1}{18}M(3c^2 + a^2)$ $I_{z_c} = \frac{1}{18}M(3b^2 + a^2)$ $I_y = \frac{1}{6}M(c^2 + 3a^2)$ $I_z = \frac{1}{6}M(b^2 + 3a^2)$	$r_{x_c}^2 = r_x^2 = \frac{1}{6}(b^2 + c^2)$ $r_{y_c}^2 = \frac{1}{18}(3c^2 + a^2)$ $r_{z_c}^2 = \frac{1}{18}(3b^2 + a^2)$ $r_y^2 = \frac{1}{6}(c^2 + 3a^2)$ $r_z^2 = \frac{1}{6}(b^2 + 3a^2)$	$I_{x_c y_c}$, etc. $= 0$ I_{xy}, etc. $= 0$

ANSWERS TO PROBLEMS

9. 14.4 lb
10. 171 lb
$\cos\theta_x = 0.581$
$\cos\theta_y = 0.421$
$\cos\theta_z = 0.698$
11. 29.2 lb
12. 13.7 lb; 324.5°
13. 38.5 lb; −26.9 lb
14. $\frac{x}{b} = 0.587$
$\frac{1}{3} < 0.587 < \frac{2}{3}$
16. −600 ft-lb
18. −52.1 ft-lb
19. $M = 1670$ in.-lb
$M_x = -1114$ in.-lb; $M_y = 568$ in.-lb
$M_z = 1108$ in.-lb
20. $M_y = -1707$ ft-lb
$M_z = 570$ ft-lb
21. $x = -3.12$
22. $x = -0.8$ ft
$z = -8.0$ ft
25. $\boldsymbol{a} + \boldsymbol{b} = 5\boldsymbol{i} + 3\boldsymbol{j} + 5\boldsymbol{k}$
$\boldsymbol{a} - \boldsymbol{b} = \boldsymbol{i} + \boldsymbol{j} + 5\boldsymbol{k}$
$\boldsymbol{a} \cdot \boldsymbol{b} = 8$
$\boldsymbol{a} \times \boldsymbol{b} = -5\boldsymbol{i} + 10\boldsymbol{j} - \boldsymbol{k}$
27. 38° 11′
28. $-10\boldsymbol{k} - 30\boldsymbol{i}$
30. $2.33\boldsymbol{e}$ ft per sec
$$\boldsymbol{e} = \frac{1.2}{2.33}\boldsymbol{i} - \frac{1.6}{2.33}\boldsymbol{j} - \frac{1.2}{2.33}\boldsymbol{k}$$
31. 54.0 ft-lb
32. −40.1 ft-lb
33. $68\boldsymbol{j} + 39\boldsymbol{k}$
34. $28.3\boldsymbol{i} - 104\boldsymbol{j} + 585\boldsymbol{k}$
35. $2Pa(-\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k})$
36. 230 lb; $F_A = 141$ lb, $F_B = 89$ lb
37. 180 lb; $x = -3.0$, $z = 1.11$
38. −7.5 lb, 20 lb, 37.5 lb
39. Magnitude of force = 38.2 lb
40. $29.6\boldsymbol{i} + 23.3\boldsymbol{j} + 22.2\boldsymbol{k}$
44. 268 lb; 1414 ft-lb
45. 160 lb; −179 ft-lb
46. $R = 231$ lb; $C = 846$ ft-lb
47. $6x_1 + 3y_1 + 2z_1 = 0$
50. 292 lb; 208 lb
51. 46° 8′
54. 424 lb
55. $D = 105$ lb (T); $E = 1455$ lb (T);
$A = 500$ lb
57. $F_A = 500$ lb; $F_B = 300$ lb
58. 45°
59. $A = 139$ lb; $B = 207$ lb
60. 30°
61. 880 lb; 471 lb
62. 35.4 lb; 46.7 lb; 67.9 lb
63. $A_x = -500$ lb; $A_y = -660$ lb;
$A_z = -154$ lb; $D = 601$ lb;
$E = 1169$ lb
64. $A_x = 300$ lb; $A_y = 100$ lb;
$A_z = -35.0$ lb; $B_y = 25$ lb;
$B_z = -35.0$ lb
65. At A, 235 lb; 450 lb; at B, 1310 lb
66. $A = 346$ lb, $B = 338$ lb
67. $x = 3.07$ ft, $A = 96.6$ lb;
$B = 114$ lb
68. $R_1 = 28.9$ lb; $R_2 = 281$ lb
69. $AB = 642$ lb (T); $AC = 783$ lb (T);
$DC = 1042$ lb (C)
70. $A_x = 414$ lb; $A_y = -410$ lb;
$B = -1594$ lb
71. $L_y = 105$ lb, $L_z = -8.11$ lb,
$R_y = 176$ lb, $R_z = 40.6$ lb,
$P = 37.5$ lb
72. $A = 420$ lb, $B = 347$ lb,
$D = -229$ lb
73. $A = 8620$ lb, $B = 7120$ lb,
$C = 3720$ lb
74. $A = 1020$ lb, $B = 993$ lb,
$F = 900$ lb, $CE = 0$, $DE = 1749$ lb
77. 212 lb
79. $\frac{AB}{AC} = \frac{HE}{HD}$

80. $\alpha = \cos^{-1}(\frac{1}{4})$
81. $-30°$
82. $F = W\dfrac{(r_2 - r_1)}{2r_2}$
84. Unstable
85. $h = \sqrt{\frac{2}{3}}\,R$
86. $\theta_1 = \cos^{-1}\left\{\dfrac{a^2\left[1 - \left(\dfrac{W_2}{W_1}\right)^2\right] + l^2}{2al}\right\}$
$\theta_2 = 0°, \theta_3 = 180°$
97. 53.4 in.^{4}
98. $x_c = 4.00, y_c = 2.84, z_c = 2.86$
99. 154 lb
100. $x_c = 2.25, y_c = 1.35, z_c = 1.80$
101. $h = 2.8$ in.
111. 27.8 lb per in., 44.5 lb per in.
112. 2250 lb
113. Max. intensity = 200 lb per ft
115. 11,500 lb
116. 759 lb per ft
117. $13.2 > \gamma > 49.2$ lb per ft^3
118. $0.45l$
119. 1.73 in.
120. $\alpha = \tan^{-1}(\frac{8}{15})$
121. 286 lb
122. $\alpha = 2\tan^{-1}\mu = 9° 10'$
124. $0.043R$
125. $a = \dfrac{h}{2\mu}$
126. Slide
127. To raise, $F = 0.348P$; self-locking
128. 100 lb (minimum)
129. 26.7 lb
130. Const. p gives approx. $\frac{1}{2}$% greater loss
131. $\dfrac{P\mu d}{3\sin\alpha}$
132. 10.8 in., 1900 lb
134. 2840 in.-lb
136. 2.35 turns
138. $\dfrac{F_1}{F_2} > e^{\frac{\mu\alpha}{\left(\sin\frac{\beta}{2} + \mu\cos\frac{\beta}{2}\right)}}$
139. 222 lb, 111 lb
141. $BD = 1500$ lb (C), $CD = 1250$ lb (C)
142. $AB = 8660$ lb (T), $FG = 5580$ lb (C)
144. F
145. 1115 lb (T), 298 lb (C)
146. Center vertical member $F = 4330$ lb (T)
148. Magnitude of resultant = 52.9 lb
149. Magnitude of resultant = 0
150. 1167 lb; 1333 lb
151. 200 lb (down); 8200 lb (up)
152. 2150 lb (right end)
153. $M_A = 720$ ft-lb; $M_B = 2820$ ft-lb
154. 5 ft from left end
159. $M_{max} = 20{,}700$ ft-lb
160. $M_{max} = -12{,}000$ ft-lb
161. $M_{max} = 22{,}800$ ft-lb
162. $M_{max} = -27{,}000$ ft-lb
163. $M_{max} = -16{,}670$ ft-lb
164. $M_{max} = 6{,}480$ ft-lb
169. 12,500 lb
170. 10,200 lb
171. 5460 lb
172. 950 ft
173. 133 ft; 2045 lb
174. 4050 lb; 1025 ft
175. 398 ft
176. 101 ft; 162 lb
180. $\frac{1}{2}$ ext. of weightless bar
181. 1.45×10^{-6} in.
182. $\dfrac{5}{9}\dfrac{Pl}{AE}$
183. $\dfrac{b}{a^2}\dfrac{Pl^2}{AE}$
184. 0.313 in.
185. $\delta_v = 0.44$ in.; $\delta_h = 0.084$ in.
186. $\delta_v = 0.63$ in.; $\delta_h = 0.056$ in.
187. 0.175 in.
188. $\frac{8}{5}P$
189. $\sigma_1 = 833$ lb per sq in.; $\sigma_2 = 2500$ lb per sq in.; $\sigma_3 = 4167$ lb per sq in.
190. $F_1 = 3260$ lb; $F_2 = 4350$ lb
191. 71%
192. $\dfrac{b}{a+b}P$
193. 20,100 lb per sq in.
194. $\sigma = 3100$ lb per sq in.

195. $F = \frac{2P}{n} \sin \theta$

196. $F_1 = 5800$ lb, $F_2 = 12{,}200$ lb, $F_3 = 5320$ lb

197. 55,000 lb; 45,000 lb

198. Linear distribution of stress

$$\sigma_{max} = 1.6 \frac{P}{a^2}$$

199. $\delta_1 = + \frac{Wl}{AE} (\sin \alpha + \mu \cos \alpha)$

$$\delta_2 = - \frac{Wl}{AE} (\sin \alpha - \mu \cos \alpha)$$

200. 4,350 lb per sq in. (Tension)
2,175 lb per sq in. (Compression)

201. 6,850 lb per sq in. (Tension)
325 lb per sq in. (Tension)

202. $\sigma_o = 23{,}600$ lb per sq in. (Comp.)
$\sigma_i = 40{,}800$ lb per sq in. (Tens.)

203. 4330 lb per sq in.; 65%

204. 9010 lb per sq in.

205. 365 lb per sq in.; 225 lb per sq in.

206. 16,600 lb per sq in.

207. 19,500 lb per sq in.

208. 4650, 700, 4650 lb per sq in.

209. +4400, −2950; +3650, −3500 lb per sq in.

210. 14,500 lb per sq in.

211. 333 lb per sq in.

INDEX